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THE COLLECTED WORKS

OF

J. WILLARD GIBBS, PH.D., LL.D.

FORMERLY PROFESSOR OF MATHEMATICAL PHYSICS IN YALE UNIVERSITY

IN TWO VOLUMES

VOLUME II

PART ONE

ELEMENTARY PRINCIPLES IN STATISTICAL MECHANICS

PART TWO

DYNAMICS

VECTOR ANALYSIS AND MULTIPLE ALGEBRA

ELECTROMAGNETIC THEORY OF LIGHT

ETC.

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PART ONE

ELEMENTARY PRINCIPLES IN STATISTICAL MECHANICS

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PREFACE.

THE usual point of view in the study of mechanics is that where the attention is mainly directed to the changes which take place in the course of time in a given system. The principal problem is the determination of the condition of the system with respect to configuration and velocities at any required time, when its condition in these respects has been given for some one time, and the fundamental equations are those which express the changes continually taking place in the system. Inquiries of this kind are often simplified by taking into consideration conditions of the system other than those through which it actually passes or is supposed to pass, but our attention is not usually carried beyond conditions differing infinitesimally from those which are regarded as actual.

For some purposes, however, it is desirable to take a broader view of the subject. We may imagine a great number of systems of the same nature, but differing in the configurations and velocities which they have at a given instant, and differing not merely infinitesimally, but it may be so as to embrace every conceivable combination of configuration and velocities. And here we may set the problem, not to follow a particular system through its succession of configurations, but to determine how the whole number of systems will be distributed among the various conceivable configurations and velocities at any required time, when the distribution has been given for some one time. The fundamental equation for this inquiry is that which gives the rate of change of the number of systems which fall within any infinitesimal limits of configuration and velocity.

Such inquiries have been called by Maxwell *statistical*. They belong to a branch of mechanics which owes its origin to the desire to explain the laws of thermodynamics on mechanical principles, and of which Clausius, Maxwell, and Boltzmann are to be regarded as the principal founders. The first inquiries in this field were indeed somewhat narrower in their scope than that which has been mentioned, being applied to the particles of a system, rather than to independent systems. Statistical inquiries were next directed to the phases (or conditions with respect to configuration and velocity) which succeed one another in a given system in the course of time. The explicit consideration of a great number of systems and their distribution in phase, and of the permanence or alteration of this distribution in the course of time is perhaps first found in Boltzmann's paper on the "Zusammenhang zwischen den Sätzen über das Verhalten mehratomiger Gasmoleküle mit Jacobi's Princip des letzten Multiplicators" (1871).

But although, as a matter of history, statistical mechanics owes its origin to investigations in thermodynamics, it seems eminently worthy of an independent development, both on account of the elegance and simplicity of its principles, and because it yields new results and places old truths in a new light in departments quite outside of thermodynamics. Moreover, the separate study of this branch of mechanics seems to afford the best foundation for the study of rational thermodynamics and molecular mechanics.

The laws of thermodynamics, as empirically determined, express the approximate and probable behavior of systems of a great number of particles, or, more precisely, they express the laws of mechanics for such systems as they appear to beings who have not the fineness of perception to enable them to appreciate quantities of the order of magnitude of those which relate to single particles, and who cannot repeat their experiments often enough to obtain any but the most probable results. The laws of statistical mechanics apply to conservative systems of any number of degrees of freedom,

and are exact. This does not make them more difficult to establish than the approximate laws for systems of a great many degrees of freedom, or for limited classes of such systems. The reverse is rather the case, for our attention is not diverted from what is essential by the peculiarities of the system considered, and we are not obliged to satisfy ourselves that the effect of the quantities and circumstances neglected will be negligible in the result. The laws of thermodynamics may be easily obtained from the principles of statistical mechanics, of which they are the incomplete expression, but they make a somewhat blind guide in our search for those laws. This is perhaps the principal cause of the slow progress of rational thermodynamics, as contrasted with the rapid deduction of the consequences of its laws as empirically established. To this must be added that the rational foundation of thermodynamics lay in a branch of mechanics of which the fundamental notions and principles, and the characteristic operations, were alike unfamiliar to students of mechanics.

We may therefore confidently believe that nothing will more conduce to the clear apprehension of the relation of thermodynamics to rational mechanics, and to the interpretation of observed phenomena with reference to their evidence respecting the molecular constitution of bodies, than the study of the fundamental notions and principles of that department of mechanics to which thermodynamics is especially related.

Moreover, we avoid the gravest difficulties when, giving up the attempt to frame hypotheses concerning the constitution of material bodies, we pursue statistical inquiries as a branch of rational mechanics. In the present state of science, it seems hardly possible to frame a dynamic theory of molecular action which shall embrace the phenomena of thermodynamics, of radiation, and of the electrical manifestations which accompany the union of atoms. Yet any theory is obviously inadequate which does not take account of all these phenomena. Even if we confine our attention to the

phenomena distinctively thermodynamic, we do not escape difficulties in as simple a matter as the number of degrees of freedom of a diatomic gas. It is well known that while theory would assign to the gas six degrees of freedom per molecule, in our experiments on specific heat we cannot account for more than five. Certainly, one is building on an insecure foundation, who rests his work on hypotheses concerning the constitution of matter.

Difficulties of this kind have deterred the author from attempting to explain the mysteries of nature, and have forced him to be contented with the more modest aim of deducing some of the more obvious propositions relating to the statistical branch of mechanics. Here, there can be no mistake in regard to the agreement of the hypotheses with the facts of nature, for nothing is assumed in that respect. The only error into which one can fall, is the want of agreement between the premises and the conclusions, and this, with care, one may hope, in the main, to avoid.

The matter of the present volume consists in large measure of results which have been obtained by the investigators mentioned above, although the point of view and the arrangement may be different. These results, given to the public one by one in the order of their discovery, have necessarily, in their original presentation, not been arranged in the most logical manner.

In the first chapter we consider the general problem which has been mentioned, and find what may be called the fundamental equation of statistical mechanics. A particular case of this equation will give the condition of statistical equilibrium, *i. e.*, the condition which the distribution of the systems in phase must satisfy in order that the distribution shall be permanent. In the general case, the fundamental equation admits an integration, which gives a principle which may be variously expressed, according to the point of view from which it is regarded, as the conservation of density-in-phase, or of extension-in-phase, or of probability of phase.

In the second chapter, we apply this principle of conservation of probability of phase to the theory of errors in the calculated phases of a system, when the determination of the arbitrary constants of the integral equations are subject to error. In this application, we do not go beyond the usual approximations. In other words, we combine the principle of conservation of probability of phase, which is exact, with those approximate relations, which it is customary to assume in the "theory of errors."

In the third chapter we apply the principle of conservation of extension-in-phase to the integration of the differential equations of motion. This gives Jacobi's "last multiplier," as has been shown by Boltzmann.

In the fourth and following chapters we return to the consideration of statistical equilibrium, and confine our attention to conservative systems. We consider especially ensembles of systems in which the index (or logarithm) of probability of phase is a linear function of the energy. This distribution, on account of its unique importance in the theory of statistical equilibrium, I have ventured to call *canonical*, and the divisor of the energy, the *modulus* of distribution. The moduli of ensembles have properties analogous to temperature, in that equality of the moduli is a condition of equilibrium with respect to exchange of energy, when such exchange is made possible.

We find a differential equation relating to average values in the ensemble which is identical in form with the fundamental differential equation of thermodynamics, the average index of probability of phase, with change of sign, corresponding to entropy, and the modulus to temperature

For the average square of the anomalies of the energy, we find an expression which vanishes in comparison with the square of the average energy, when the number of degrees of freedom is indefinitely increased. An ensemble of systems in which the number of degrees of freedom is of the same order of magnitude as the number of molecules in the bodies

with which we experiment, if distributed canonically, would therefore appear to human observation as an ensemble of systems in which all have the same energy.

We meet with other quantities, in the development of the subject, which, when the number of degrees of freedom is very great, coincide sensibly with the modulus, and with the average index of probability, taken negatively, in a canonical ensemble, and which, therefore, may also be regarded as corresponding to temperature and entropy. The correspondence is however imperfect, when the number of degrees of freedom is not very great, and there is nothing to recommend these quantities except that in definition they may be regarded as more simple than those which have been mentioned. In Chapter XIV, this subject of thermodynamic analogies is discussed somewhat at length.

Finally, in Chapter XV, we consider the modification of the preceding results which is necessary when we consider systems composed of a number of entirely similar particles, or, it may be, of a number of particles of several kinds, all of each kind being entirely similar to each other, and when one of the variations to be considered is that of the numbers of the particles of the various kinds which are contained in a system. This supposition would naturally have been introduced earlier, if our object had been simply the expression of the laws of nature. It seemed desirable, however, to separate sharply the purely thermodynamic laws from those special modifications which belong rather to the theory of the properties of matter.

J. W. G.

New Haven, December, 1901.

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ELEMENTARY PRINCIPLES IN STATISTICAL MECHANICS

ELEMENTARY PRINCIPLES IN STATISTICAL MECHANICS

CHAPTER I.

GENERAL NOTIONS. THE PRINCIPLE OF CONSERVATION OF EXTENSION-IN-PHASE.

We shall use Hamilton's form of the equations of motion for a system of n degrees of freedom, writing $q_1, \ldots q_n$ for the (generalized) coördinates, $\dot{q}_1, \ldots \dot{q}_n$ for the (generalized) velocities, and

$$F_1\,dq_1 + F_2\,dq_2 \ldots + F_n\,dq_n \qquad (1)$$

for the moment of the forces. We shall call the quantities $F_1, \ldots F_n$ the (generalized) forces, and the quantities $p_1 \ldots p_n$, defined by the equations

$$p_1 = \frac{d\epsilon_p}{d\dot{q}_1}, \quad p_2 = \frac{d\epsilon_p}{d\dot{q}_2}, \quad \text{etc.}, \qquad (2)$$

where ϵ_p denotes the kinetic energy of the system, the (generalized) momenta. The kinetic energy is here regarded as a function of the velocities and coördinates. We shall usually regard it as a function of the momenta and coördinates,* and on this account we denote it by ϵ_p. This will not prevent us from occasionally using formulæ like (2), where it is sufficiently evident the kinetic energy is regarded as function of the $\dot{q}$'s and q's. But in expressions like $d\epsilon_p/dq_1$, where the denominator does not determine the question, the kinetic

* The use of the momenta instead of the velocities as independent variables is the characteristic of Hamilton's method which gives his equations of motion their remarkable degree of simplicity. We shall find that the fundamental notions of statistical mechanics are most easily defined, and are expressed in the most simple form, when the momenta with the coördinates are used to describe the state of a system.

energy is always to be treated in the differentiation as function of the p's and q's.

We have then

$$\dot{q}_1 = \frac{d\epsilon_p}{dp_1}, \quad \dot{p}_1 = -\frac{d\epsilon_p}{dq_1} + F_1, \quad \text{etc.} \tag{3}$$

These equations will hold for any forces whatever. If the forces are conservative, in other words, if the expression (1) is an exact differential, we may set

$$F_1 = -\frac{d\epsilon_q}{dq_1}, \quad F_2 = -\frac{d\epsilon_q}{dq_2}, \quad \text{etc.}, \tag{4}$$

where ϵ_q is a function of the coördinates which we shall call the potential energy of the system. If we write ϵ for the total energy, we shall have

$$\epsilon = \epsilon_p + \epsilon_q, \tag{5}$$

and equations (3) may be written

$$\dot{q}_1 = \frac{d\epsilon}{dp_1}, \quad \dot{p}_1 = -\frac{d\epsilon}{dq_1}, \quad \text{etc.} \tag{6}$$

The potential energy (ϵ_q) may depend on other variables beside the coördinates $q_1 \ldots q_n$. We shall often suppose it to depend in part on coördinates of external bodies, which we shall denote by a_1, a_2, etc. We shall then have for the complete value of the differential of the potential energy *

$$d\epsilon_q = -F_1\,dq_1 \ldots - F_n\,dq_n - A_1\,da_1 - A_2\,da_2 - \text{etc.}, \tag{7}$$

where A_1, A_2, etc., represent forces (in the generalized sense) exerted by the system on external bodies. For the total energy (ϵ) we shall have

$$d\epsilon = \dot{q}_1\,dp_1 \ldots + \dot{q}_n\,dp_n - \dot{p}_1\,dq_1 \ldots - \dot{p}_n\,dq_n - A_1\,da_1 - A_2\,da_2 - \text{etc.} \tag{8}$$

It will be observed that the kinetic energy (ϵ_p) in the most general case is a quadratic function of the p's (or $\dot{q}$'s)

* It will be observed, that although we call ϵ_q the potential energy of the system which we are considering, it is really so defined as to include that energy which might be described as mutual to that system and external bodies.

involving also the q's but not the a's; that the potential energy, when it exists, is function of the q's and a's; and that the total energy, when it exists, is function of the p's (or $\dot{q}$'s), the q's, and the a's. In expressions like $d\epsilon/dq_1$, the p's, and not the $\dot{q}$'s, are to be taken as independent variables, as has already been stated with respect to the kinetic energy.

Let us imagine a great number of independent systems, identical in nature, but differing in phase, that is, in their condition with respect to configuration and velocity. The forces are supposed to be determined for every system by the same law, being functions of the coördinates of the system $q_1, \ldots q_n$, either alone or with the coördinates a_1, a_2, etc. of certain external bodies. It is not necessary that they should be derivable from a force-function. The external coördinates a_1, a_2, etc. may vary with the time, but at any given time have fixed values. In this they differ from the internal coördinates $q_1, \ldots q_n$, which at the same time have different values in the different systems considered.

Let us especially consider the number of systems which at a given instant fall within specified limits of phase, viz., those for which

$$\left.\begin{array}{ll} p_1' < p_1 < p_1'', & q_1' < q_1 < q_1'', \\ p_2' < p_2 < p_2'', & q_2' < q_2 < q_2'', \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ p_n' < p_n < p_n'', & q_n' < q_n < q_n'', \end{array}\right\} \qquad (9)$$

the accented letters denoting constants. We shall suppose the differences $p_1'' - p_1'$, $q_1'' - q_1'$, etc. to be infinitesimal, and that the systems are distributed in phase in some continuous manner,* so that the number having phases within the limits specified may be represented by

$$D\,(p_1'' - p_1')\ldots(p_n'' - p_n')\,(q_1'' - q_1')\ldots(q_n'' - q_n'), \qquad (10)$$

* In strictness, a finite number of systems cannot be distributed continuously in phase. But by increasing indefinitely the number of systems, we may approximate to a continuous law of distribution, such as is here described. To avoid tedious circumlocution, language like the above may be allowed, although wanting in precision of expression, when the sense in which it is to be taken appears sufficiently clear.

or more briefly by

$$D\, dp_1 \ldots dp_n\, dq_1 \ldots dq_n, \tag{11}$$

where D is a function of the p's and q's and in general of t also, for as time goes on, and the individual systems change their phases, the distribution of the ensemble in phase will in general vary. In special cases, the distribution in phase will remain unchanged. These are cases of *statistical equilibrium.*

If we regard all possible phases as forming a sort of extension of $2n$ dimensions, we may regard the product of differentials in (11) as expressing an element of this extension, and D as expressing the density of the systems in that element. We shall call the product

$$dp_1 \ldots dp_n\, dq_1 \ldots dq_n \tag{12}$$

an element of *extension-in-phase*, and D the *density-in-phase* of the systems.

It is evident that the changes which take place in the density of the systems in any given element of extension-in-phase will depend on the dynamical nature of the systems and their distribution in phase at the time considered.

In the case of conservative systems, with which we shall be principally concerned, their dynamical nature is completely determined by the function which expresses the energy (ϵ) in terms of the p's, q's, and a's (a function supposed identical for all the systems); in the more general case which we are considering, the dynamical nature of the systems is determined by the functions which express the kinetic energy (ϵ_p) in terms of the p's and q's, and the forces in terms of the q's and a's. The distribution in phase is expressed for the time considered by D as function of the p's and q's. To find the value of dD/dt for the specified element of extension-in-phase, we observe that the number of systems within the limits can only be varied by systems passing the limits, which may take place in $4n$ different ways, viz., by the p_1 of a system passing the limit p_1', or the limit p_1'', or by the q_1 of a system passing the limit q_1', or the limit q_1'', etc. Let us consider these cases separately.

In the first place, let us consider the number of systems which in the time dt pass into or out of the specified element by p_1 passing the limit p_1'. It will be convenient, and it is evidently allowable, to suppose dt so small that the quantities $\dot{p}_1\,dt$, $\dot{q}_1\,dt$, etc., which represent the increments of p_1, q_1, etc., in the time dt shall be infinitely small in comparison with the infinitesimal differences $p_1'' - p_1'$, $q_1'' - q_1'$, etc., which determine the magnitude of the element of extension-in-phase. The systems for which p_1 passes the limit p_1' in the interval dt are those for which at the commencement of this interval the value of p_1 lies between p_1' and $p_1' - \dot{p}_1\,dt$, as is evident if we consider separately the cases in which $\dot{p}_1$ is positive and negative. Those systems for which p_1 lies between these limits, and the other p's and q's between the limits specified in (9), will therefore pass into or out of the element considered according as $\dot{p}$ is positive or negative, unless indeed they also pass some other limit specified in (9) during the same interval of time. But the number which pass any two of these limits will be represented by an expression containing the square of dt as a factor, and is evidently negligible, when dt is sufficiently small, compared with the number which we are seeking to evaluate, and which (with neglect of terms containing dt^2) may be found by substituting $\dot{p}_1\,dt$ for $p_1'' - p_1'$ in (10) or for dp_1 in (11).

The expression

$$D\dot{p}_1\,dt\,dp_2 \ldots dp_n\,dq_1 \ldots dq_n \qquad (13)$$

will therefore represent, according as it is positive or negative, the increase or decrease of the number of systems within the given limits which is due to systems passing the limit p_1'. A similar expression, in which however D and $\dot{p}$ will have slightly different values (being determined for p_1'' instead of p_1'), will represent the decrease or increase of the number of systems due to the passing of the limit p_1''. The difference of the two expressions, or

$$\frac{d(D\dot{p}_1)}{dp_1}\,dp_1 \ldots dp_n\,dq_1 \ldots dq_n\,dt \qquad (14)$$

will represent algebraically the decrease of the number of systems within the limits due to systems passing the limits p_1' and p_1''.

The decrease in the number of systems within the limits due to systems passing the limits q_1' and q_1'' may be found in the same way. This will give

$$\left(\frac{d(D\dot{p}_1)}{dp_1} + \frac{d(D\dot{q}_1)}{dq_1}\right) dp_1 \ldots dp_n\, dq_1 \ldots dq_n\, dt \qquad (15)$$

for the decrease due to passing the four limits p_1', p_1'', q_1', q_1''. But since the equations of motion (3) give

$$\frac{d\dot{p}_1}{dp_1} + \frac{d\dot{q}_1}{dq_1} = 0, \qquad (16)$$

the expression reduces to

$$\left(\frac{dD}{dp_1}\dot{p}_1 + \frac{dD}{dq_1}\dot{q}_1\right) dp_1 \ldots dp_n\, dq_1 \ldots dq_n\, dt. \qquad (17)$$

If we prefix Σ to denote summation relative to the suffixes $1 \ldots n$, we get the total decrease in the number of systems within the limits in the time dt. That is,

$$\Sigma\left(\frac{dD}{dp_1}\dot{p}_1 + \frac{dD}{dq_1}\dot{q}_1\right) dp_1 \ldots dp_n\, dq_1 \ldots dq_n\, dt = - dD\, dp_1 \ldots dp_n\, dq_1 \ldots dq_n, \qquad (18)$$

or

$$\left(\frac{dD}{dt}\right)_{p,q} = -\Sigma\left(\frac{dD}{dp_1}\dot{p}_1 + \frac{dD}{dq_1}\dot{q}_1\right), \qquad (19)$$

where the suffix applied to the differential coefficient indicates that the p's and q's are to be regarded as constant in the differentiation. The condition of statistical equilibrium is therefore

$$\Sigma\left(\frac{dD}{dp_1}\dot{p}_1 + \frac{dD}{dq_1}\dot{q}_1\right) = 0. \qquad (20)$$

If at any instant this condition is fulfilled for all values of the p's and q's, $(dD/dt)_{p,q}$ vanishes, and therefore the condition will continue to hold, and the distribution in phase will be permanent, so long as the external coördinates remain constant. But the statistical equilibrium would in general be disturbed by a change in the values of the external coördinates, which

would alter the values of the $\dot{p}$'s as determined by equations (3), and thus disturb the relation expressed in the last equation.

If we write equation (19) in the form

$$\left(\frac{dD}{dt}\right)_{p,q} dt + \Sigma \left(\frac{dD}{dp_1} \dot{p}_1 \, dt + \frac{dD}{dq_1} \dot{q}_1 \, dt\right) = 0, \tag{21}$$

it will be seen to express a theorem of remarkable simplicity. Since D is a function of t, $p_1, \ldots p_n$, $q_1, \ldots q_n$, its complete differential will consist of parts due to the variations of all these quantities. Now the first term of the equation represents the increment of D due to an increment of t (with constant values of the p's and q's), and the rest of the first member represents the increments of D due to increments of the p's and q's, expressed by $\dot{p}_1 \, dt$, $\dot{q}_1 \, dt$, etc. But these are precisely the increments which the p's and q's receive in the movement of a system in the time dt. The whole expression represents the total increment of D for the varying phase of a moving system. We have therefore the theorem: —

*In an ensemble of mechanical systems identical in nature and subject to forces determined by identical laws, but distributed in phase in any continuous manner, the density-in-phase is constant in time for the varying phases of a moving system; provided, that the forces of a system are functions of its coordinates, either alone or with the time.**

This may be called the principle of *conservation of density-in-phase.* It may also be written

$$\left(\frac{dD}{dt}\right)_{a,\ldots h} = 0, \tag{22}$$

where $a, \ldots h$ represent the arbitrary constants of the integral equations of motion, and are suffixed to the differential co-

* The condition that the forces $F_1, \ldots F_n$ are functions of $q_1, \ldots q_n$ and a_1, a_2, etc., which last are functions of the time, is analytically equivalent to the condition that $F_1, \ldots F_n$ are functions of $q_1, \ldots q_n$ and the time. Explicit mention of the external coördinates, a_1, a_2, etc., has been made in the preceding pages, because our purpose will require us hereafter to consider these coördinates and the connected forces, A_1, A_2, etc., which represent the action of the systems on external bodies.

efficient to indicate that they are to be regarded as constant in the differentiation.

We may give to this principle a slightly different expression. Let us call the value of the integral

$$\int \cdots \int dp_1 \cdots dp_n \, dq_1 \cdots dq_n \tag{23}$$

taken within any limits the *extension-in-phase* within those limits.

When the phases bounding an extension-in-phase vary in the course of time according to the dynamical laws of a system subject to forces which are functions of the coördinates either alone or with the time, the value of the extension-in-phase thus bounded remains constant. In this form the principle may be called the principle of *conservation of extension-in-phase.* In some respects this may be regarded as the most simple statement of the principle, since it contains no explicit reference to an ensemble of systems.

Since any extension-in-phase may be divided into infinitesimal portions, it is only necessary to prove the principle for an infinitely small extension. The number of systems of an ensemble which fall within the extension will be represented by the integral

$$\int \cdots \int D \, dp_1 \cdots dp_n \, dq_1 \cdots dq_n.$$

If the extension is infinitely small, we may regard D as constant in the extension and write

$$D \int \cdots \int dp_1 \cdots dp_n \, dq_1 \cdots dq_n$$

for the number of systems. The value of this expression must be constant in time, since no systems are supposed to be created or destroyed, and none can pass the limits, because the motion of the limits is identical with that of the systems. But we have seen that D is constant in time, and therefore the integral

$$\int \cdots \int dp_1 \cdots dp_n \, dq_1 \cdots dq_n,$$

which we have called the extension-in-phase, is also constant in time.*

Since the system of coördinates employed in the foregoing discussion is entirely arbitrary, the values of the coördinates relating to any configuration and its immediate vicinity do not impose any restriction upon the values relating to other configurations. The fact that the quantity which we have called density-in-phase is constant in time for any given system, implies therefore that its value is independent of the coördinates which are used in its evaluation. For let the density-in-phase as evaluated for the same time and phase by one system of coördinates be D_1', and by another system D_2'. A system which at that time has that phase will at another time have another phase. Let the density as calculated for this second time and phase by a third system of coördinates be D_3''. Now we may imagine a system of coördinates which at and near the first configuration will coincide with the first system of coördinates, and at and near the second configuration will coincide with the third system of coördinates. This will give $D_1' = D_3''$. Again we may imagine a system of coördinates which at and near the first configuration will coincide with the second system of coördinates, and at and near the

* If we regard a phase as represented by a point in space of $2n$ dimensions, the changes which take place in the course of time in our ensemble of systems will be represented by a current in such space. This current will be steady so long as the external coördinates are not varied. In any case the current will satisfy a law which in its various expressions is analogous to the hydrodynamic law which may be expressed by the phrases *conservation of volumes* or *conservation of density about a moving point*, or by the equation

$$\frac{d\dot{x}}{dx} + \frac{d\dot{y}}{dy} + \frac{d\dot{z}}{dz} = 0.$$

The analogue in statistical mechanics of this equation, viz.,

$$\frac{d\dot{p}_1}{dp_1} + \frac{d\dot{q}_1}{dq_1} + \frac{d\dot{p}_2}{dp_2} + \frac{d\dot{q}_2}{dq_2} + \text{etc.} = 0,$$

may be derived directly from equations (3) or (6), and may suggest such theorems as have been enunciated, if indeed it is not regarded as making them intuitively evident. The somewhat lengthy demonstrations given above will at least serve to give precision to the notions involved, and familiarity with their use.

second configuration will coincide with the third system of coördinates. This will give $D_2' = D_3''$. We have therefore $D_1' = D_2'$.

It follows, or it may be proved in the same way, that the value of an extension-in-phase is independent of the system of coördinates which is used in its evaluation. This may easily be verified directly. If $q_1, \dots q_n$, $Q_1, \dots Q_n$ are two systems of coördinates, and $p_1, \dots p_n$, $P_1, \dots P_n$ the corresponding momenta, we have to prove that

$$\int \dots \int dp_1 \dots dp_n dq_1 \dots dq_n = \int \dots \int dP_1 \dots dP_n dQ_1 \dots dQ_n, \quad (24)$$

when the multiple integrals are taken within limits consisting of the same phases. And this will be evident from the principle on which we change the variables in a multiple integral, if we prove that

$$\frac{d(P_1, \dots P_n, Q_1, \dots Q_n)}{d(p_1, \dots p_n, q_1, \dots q_n)} = 1, \quad (25)$$

where the first member of the equation represents a Jacobian or functional determinant. Since all its elements of the form dQ/dp are equal to zero, the determinant reduces to a product of two, and we have to prove that

$$\frac{d(P_1, \dots P_n)}{d(p_1, \dots p_n)} \frac{d(Q_1, \dots Q_n)}{d(q_1, \dots q_n)} = 1. \quad (26)$$

We may transform any element of the first of these determinants as follows. By equations (2) and (3), and in view of the fact that the $\dot{Q}$'s are linear functions of the $\dot{q}$'s and therefore of the p's, with coefficients involving the q's, so that a differential coefficient of the form $d\dot{Q}_r/dp_y$ is function of the q's alone, we get*

* The form of the equation

$$\frac{d}{dp_y} \frac{d\epsilon_p}{d\dot{Q}_x} = \frac{d}{d\dot{Q}_x} \frac{d\epsilon_p}{dp_y}$$

in (27) reminds us of the fundamental identity in the differential calculus relating to the order of differentiation with respect to independent variables. But it will be observed that here the variables $\dot{Q}_x$ and p_y are *not* independent and that the proof depends on the *linear* relation between the $\dot{Q}$'s and the p's.

$$\frac{dP_x}{dp_y} = \frac{d}{dp_y}\frac{d\epsilon_p}{d\dot{Q}_x} = \sum_{r=1}^{r=n}\left(\frac{d^2\epsilon_p}{d\dot{Q}_r\, d\dot{Q}_x}\frac{d\dot{Q}_r}{dp_y}\right) =$$

$$\frac{d}{d\dot{Q}_x}\sum_{r=1}^{r=n}\left(\frac{d\epsilon_p}{d\dot{Q}_r}\frac{d\dot{Q}_r}{dp_y}\right) = \frac{d}{d\dot{Q}_x}\frac{d\epsilon_p}{dp_y} = \frac{d\dot{q}_y}{d\dot{Q}_x}. \quad (27)$$

But since

$$\dot{q}_y = \sum_{r=1}^{r=n}\left(\frac{dq_y}{dQ_r}\dot{Q}_r\right),$$

$$\frac{d\dot{q}_y}{d\dot{Q}_x} = \frac{dq_y}{dQ_x}. \quad (28)$$

Therefore,

$$\frac{d(P_1, \ldots P_n)}{d(p_1, \ldots p_n)} = \frac{d(\dot{q}_1, \ldots \dot{q}_n)}{d(\dot{Q}_1, \ldots \dot{Q}_n)} = \frac{d(q_1, \ldots q_n)}{d(Q_1, \ldots Q_n)}. \quad (29)$$

The equation to be proved is thus reduced to

$$\frac{d(q_1, \ldots q_n)}{d(Q_1, \ldots Q_n)}\frac{d(Q_1, \ldots Q_n)}{d(q_1, \ldots q_n)} = 1, \quad (30)$$

which is easily proved by the ordinary rule for the multiplication of determinants.

The numerical value of an extension-in-phase will however depend on the units in which we measure energy and time. For a product of the form $dp\, dq$ has the dimensions of energy multiplied by time, as appears from equation (2), by which the momenta are defined. Hence an extension-in-phase has the dimensions of the nth power of the product of energy and time. In other words, it has the dimensions of the nth power of *action*, as the term is used in the 'principle of *Least Action*.'

If we distinguish by accents the values of the momenta and coördinates which belong to a time t', the unaccented letters relating to the time t, the principle of the conservation of extension-in-phase may be written

$$\int\ldots\int dp_1 \ldots dp_n dq_1 \ldots dq_n = \int\ldots\int dp_1' \ldots dp_n' dq_1' \ldots dq_n', \quad (31)$$

or more briefly

$$\int\ldots\int dp_1 \ldots dq_n = \int\ldots\int dp_1' \ldots dq_n', \quad (32)$$

the limiting phases being those which belong to the same systems at the times t and t' respectively. But we have identically

$$\int \cdots \int dp_1 \cdots dq_n = \int \cdots \int \frac{d(p_1 \cdots q_n)}{d(p_1' \cdots q_n')} dp_1' \cdots dq_n'$$

for such limits. The principle of conservation of extension-in-phase may therefore be expressed in the form

$$\frac{d(p_1, \cdots q_n)}{d(p_1' \cdots q_n')} = 1. \tag{33}$$

This equation is easily proved directly. For we have identically

$$\frac{d(p_1, \cdots q_n)}{d(p_1', \cdots q_n)} = \frac{d(p_1, \cdots q_n)}{d(p_1'', \cdots q_n'')} \frac{d(p_1'', \cdots q_n'')}{d(p_1', \cdots q_n')},$$

where the double accents distinguish the values of the momenta and coördinates for a time t''. If we vary t, while t' and t'' remain constant, we have

$$\frac{d}{dt} \frac{d(p_1, \cdots q_n)}{d(p_1', \cdots q_n')} = \frac{d(p_1'', \cdots q_n'')}{d(p_1', \cdots q_n')} \frac{d}{dt} \frac{d(p_1, \cdots q_n)}{d(p_1'', \cdots q_n'')}. \tag{34}$$

Now since the time t'' is entirely arbitrary, nothing prevents us from making t'' identical with t at the moment considered. Then the determinant

$$\frac{d(p_1, \cdots q_n)}{d(p_1'', \cdots q_n'')}$$

will have unity for each of the elements on the principal diagonal, and zero for all the other elements. Since every term of the determinant except the product of the elements on the principal diagonal will have two zero factors, the differential of the determinant will reduce to that of the product of these elements, *i. e.*, to the sum of the differentials of these elements. This gives the equation

$$\frac{d}{dt} \frac{d(p_1, \cdots q_n)}{d(p_1'', \cdots q_n'')} = \frac{d\dot{p}_1}{dp_1''} \cdots + \frac{d\dot{p}_n}{dp_n''} + \frac{d\dot{q}_1}{dq_1''} \cdots + \frac{d\dot{q}_n}{dq_n''}.$$

Now since $t = t''$, the double accents in the second member of this equation may evidently be neglected. This will give, in virtue of such relations as (16),

$$\frac{d}{dt}\frac{d(p_1, \ldots q_n)}{d(p_1'', \ldots q_n'')} = 0,$$

which substituted in (34) will give

$$\frac{d}{dt}\frac{d(p_1, \ldots q_n)}{d(p_1', \ldots q_n')} = 0.$$

The determinant in this equation is therefore a constant, the value of which may be determined at the instant when $t = t'$, when it is evidently unity. Equation (33) is therefore demonstrated.

Again, if we write $a, \ldots h$ for a system of $2n$ arbitrary constants of the integral equations of motion, p_1, q_1, etc. will be functions of $a, \ldots h$, and t, and we may express an extension-in-phase in the form

$$\int \ldots \int \frac{d(p_1, \ldots q_n)}{d(a, \ldots h)}\, da \ldots dh. \tag{35}$$

If we suppose the limits specified by values of $a, \ldots h$, a system initially at the limits will remain at the limits. The principle of conservation of extension-in-phase requires that an extension thus bounded shall have a constant value. This requires that the determinant under the integral sign shall be constant, which may be written

$$\frac{d}{dt}\frac{d(p_1, \ldots q_n)}{d(a, \ldots h)} = 0. \tag{36}$$

This equation, which may be regarded as expressing the principle of conservation of extension-in-phase, may be derived directly from the identity

$$\frac{d(p_1, \ldots q_n)}{d(a, \ldots h)} = \frac{d(p_1, \ldots q_n)}{d(p_1', \ldots q_n')}\,\frac{d(p_1', \ldots q_n')}{d(a, \ldots h)}$$

in connection with equation (33).

Since the coördinates and momenta are functions of $a, \ldots h$, and t, the determinant in (36) must be a function of the same variables, and since it does not vary with the time, it must be a function of $a, \ldots h$ alone. We have therefore

$$\frac{d(p_1, \ldots q_n)}{d(a, \ldots h)} = \text{func. } (a, \ldots h). \tag{37}$$

It is the relative numbers of systems which fall within different limits, rather than the absolute numbers, with which we are most concerned. It is indeed only with regard to relative numbers that such discussions as the preceding will apply with literal precision, since the nature of our reasoning implies that the number of systems in the smallest element of space which we consider is very great. This is evidently inconsistent with a finite value of the total number of systems, or of the density-in-phase. Now if the value of D is infinite, we cannot speak of any definite number of systems within any finite limits, since all such numbers are infinite. But the ratios of these infinite numbers may be perfectly definite. If we write N for the total number of systems, and set

$$P = \frac{D}{N}, \tag{38}$$

P may remain finite, when N and D become infinite. The integral

$$\int \dots \int P \, dp_1 \dots dq_n \tag{39}$$

taken within any given limits, will evidently express the ratio of the number of systems falling within those limits to the whole number of systems. This is the same thing as the *probability* that an unspecified system of the ensemble (*i. e.* one of which we only know that it belongs to the ensemble) will lie within the given limits. The product

$$P \, dp_1 \dots dq_n \tag{40}$$

expresses the probability that an unspecified system of the ensemble will be found in the element of extension-in-phase $dp_1 \dots dq_n$. We shall call P the *coefficient of probability* of the phase considered. Its natural logarithm we shall call the *index of probability* of the phase, and denote it by the letter η.

If we substitute NP and Ne^{η} for D in equation (19), we get

$$\left(\frac{dP}{dt}\right)_{p,q} = -\Sigma\left(\frac{dP}{dp_1}\dot{p}_1 + \frac{dP}{dq_1}\dot{q}_1\right), \tag{41}$$

and

$$\left(\frac{d\eta}{dt}\right)_{p,q} = -\Sigma\left(\frac{d\eta}{dp_1}\dot{p}_1 + \frac{d\eta}{dq_1}\dot{q}_1\right). \tag{42}$$

The condition of statistical equilibrium may be expressed by equating to zero the second member of either of these equations.

The same substitutions in (22) give

$$\left(\frac{dP}{dt}\right)_{a,\ldots h} = 0, \tag{43}$$

and

$$\left(\frac{d\eta}{dt}\right)_{a,\ldots h} = 0. \tag{44}$$

That is, the values of P and η, like those of D, are constant in time for moving systems of the ensemble. From this point of view, the principle which otherwise regarded has been called the principle of conservation of density-in-phase or conservation of extension-in-phase, may be called the principle of conservation of the coefficient (or index) of probability of a phase varying according to dynamical laws, or more briefly, the principle of *conservation of probability of phase.* It is subject to the limitation that the forces must be functions of the coördinates of the system either alone or with the time.

The application of this principle is not limited to cases in which there is a formal and explicit reference to an ensemble of systems. Yet the conception of such an ensemble may serve to give precision to notions of probability. It is in fact customary in the discussion of probabilities to describe anything which is imperfectly known as something taken at random from a great number of things which are completely described. But if we prefer to avoid any reference to an ensemble of systems, we may observe that the probability that the phase of a system falls within certain limits at a certain time, is equal to the probability that at some other time the phase will fall within the limits formed by phases corresponding to the first. For either occurrence necessitates the other. That is, if we write P' for the coefficient of probability of the phase $p_1', \ldots q_n'$ at the time t', and P'' for that of the phase $p_1'', \ldots q_n''$ at the time t'',

$$\int \dots \int P' \, dq_1' \dots dq_n' = \int \dots \int P'' \, dp_1'' \dots dq_n'', \quad (45)$$

where the limits in the two cases are formed by corresponding phases. When the integrations cover infinitely small variations of the momenta and coördinates, we may regard P' and P'' as constant in the integrations and write

$$P' \int \dots \int dp_1' \dots dq_n'' = P' \int \dots \int dp_1'' \dots dq_n''.$$

Now the principle of the conservation of extension-in-phase, which has been proved (viz., in the second demonstration given above) independently of any reference to an ensemble of systems, requires that the values of the multiple integrals in this equation shall be equal. This gives

$$P'' = P'.$$

With reference to an important class of cases this principle may be enunciated as follows.

When the differential equations of motion are exactly known, but the constants of the integral equations imperfectly determined, the coefficient of probability of any phase at any time is equal to the coefficient of probability of the corresponding phase at any other time. By corresponding phases are meant those which are calculated for different times from the same values of the arbitrary constants of the integral equations.

Since the sum of the probabilities of all possible cases is necessarily unity, it is evident that we must have

$$\int_{\text{phases}}^{\text{all}} \dots \int P \, dp_1 \dots dq_n = 1, \quad (46)$$

where the integration extends over all phases. This is indeed only a different form of the equation

$$N = \int_{\text{phases}}^{\text{all}} \dots \int D \, dp_1 \dots dq_n,$$

which we may regard as defining N.

The values of the coefficient and index of probability of phase, like that of the density-in-phase, are independent of the system of coördinates which is employed to express the distribution in phase of a given ensemble.

In dimensions, the coefficient of probability is the reciprocal of an extension-in-phase, that is, the reciprocal of the nth power of the product of time and energy. The index of probability is therefore affected by an additive constant when we change our units of time and energy. If the unit of time is multiplied by c_t and the unit of energy is multiplied by c_ϵ, all indices of probability relating to systems of n degrees of freedom will be increased by the addition of

$$n \log c_t + n \log c_\epsilon. \qquad (47)$$

CHAPTER II.

APPLICATION OF THE PRINCIPLE OF CONSERVATION OF EXTENSION-IN-PHASE TO THE THEORY OF ERRORS.

LET us now proceed to combine the principle which has been demonstrated in the preceding chapter and which in its different applications and regarded from different points of view has been variously designated as the conservation of density-in-phase, or of extension-in-phase, or of probability of phase, with those approximate relations which are generally used in the 'theory of errors.'

We suppose that the differential equations of the motion of a system are exactly known, but that the constants of the integral equations are only approximately determined. It is evident that the probability that the momenta and coördinates at the time t' fall between the limits p_1' and $p_1' + dp_1'$, q_1' and $q_1' + dq_1'$, etc., may be expressed by the formula

$$e^{\eta'} dp_1' \dots dq_n', \tag{48}$$

where η' (the index of probability for the phase in question) is a function of the coördinates and momenta and of the time.

Let Q_1', P_1', etc. be the values of the coördinates and momenta which give the maximum value to η', and let the general value of η' be developed by Taylor's theorem according to ascending powers and products of the differences $p_1' - P_1'$, $q_1' - Q_1'$, etc., and let us suppose that we have a sufficient approximation without going beyond terms of the second degree in these differences. We may therefore set

$$\eta' = c - F', \tag{49}$$

where c is independent of the differences $p_1' - P_1'$, $q_1' - Q_1'$, etc., and F' is a homogeneous quadratic function of these

differences. The terms of the first degree vanish in virtue of the maximum condition, which also requires that F' must have a positive value except when all the differences mentioned vanish. If we set

$$C = e^{c}, \tag{50}$$

we may write for the probability that the phase lies within the limits considered

$$Ce^{-F'}\, dp_1' \ldots dq_n'. \tag{51}$$

C is evidently the maximum value of the coefficient of probability at the time considered.

In regard to the degree of approximation represented by these formulæ, it is to be observed that we suppose, as is usual in the 'theory of errors,' that the determination (explicit or implicit) of the constants of motion is of such precision that the coefficient of probability $e^{\eta'}$ or $Ce^{-F'}$ is practically zero except for very small values of the differences $p_1' - P_1'$, $q_1' - Q_1'$, etc. For very small values of these differences the approximation is evidently in general sufficient, for larger values of these differences the value of $Ce^{-F'}$ will be sensibly zero, as it should be, and in this sense the formula will represent the facts.

We shall suppose that the forces to which the system is subject are functions of the coördinates either alone or with the time. The principle of conservation of probability of phase will therefore apply, which requires that at any other time (t'') the maximum value of the coefficient of probability shall be the same as at the time t', and that the phase (P_1'', Q_1'', etc.) which has this greatest probability-coefficient, shall be that which corresponds to the phase (P_1', Q_1', etc.), *i. e.*, which is calculated from the same values of the constants of the integral equations of motion.

We may therefore write for the probability that the phase at the time t'' falls within the limits p_1'' and $p_1'' + dp_1''$, q_1'' and $q_1'' + dq_1''$, etc.,

$$Ce^{-F''}\, dp_1'' \ldots dq_n'', \tag{52}$$

where C represents the same value as in the preceding formula, viz., the constant value of the maximum coefficient of probability, and F'' is a quadratic function of the differences $p_1'' - P_1''$, $q_1'' - Q_1''$, etc., the phase (P_1'', Q_1'' etc.) being that which at the time t'' corresponds to the phase (P_1', Q_1', etc.) at the time t'.

Now we have necessarily

$$\int \dots \int Ce^{-F'} dp_1' \dots dq_n' = \int \dots \int Ce^{-F''} dp_1'' \dots dq_n'' = 1, \quad (53)$$

when the integration is extended over all possible phases. It will be allowable to set $\pm \infty$ for the limits of all the coördinates and momenta, not because these values represent the actual limits of possible phases, but because the portions of the integrals lying outside of the limits of all possible phases will have sensibly the value zero. With $\pm \infty$ for limits, the equation gives

$$\frac{C\pi^n}{\sqrt{f'}} = \frac{C\pi^n}{\sqrt{f''}} = 1, \quad (54)$$

where f' is the discriminant * of F', and f'' that of F''. This discriminant is therefore constant in time, and like C an absolute invariant in respect to the system of coördinates which may be employed. In dimensions, like C^2, it is the reciprocal of the 2nth power of the product of energy and time.

Let us see precisely how the functions F' and F'' are related. The principle of the conservation of the probability-coefficient requires that any values of the coördinates and momenta at the time t' shall give the function F' the same value as the corresponding coördinates and momenta at the time t'' give to F''. Therefore F'' may be derived from F' by substituting for $p_1', \dots q_n'$ their values in terms of $p_1'', \dots q_1''$. Now we have approximately

* This term is used to denote the determinant having for elements on the principal diagonal the coefficients of the squares in the quadratic function F', and for its other elements the halves of the coefficients of the products in F'.

$$\left.\begin{aligned} p_1' - P_1' &= \frac{dP_1'}{dP_1''}(p_1'' - P_1'') \ldots + \frac{dP_1'}{dQ_n''}(q_n'' - Q_n'') \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ q_n' - Q_n' &= \frac{dQ_n'}{dP_1''}(p_1'' - P_1'') \ldots + \frac{dQ_n'}{dQ_n''}(q_n'' - Q_n''), \end{aligned}\right\} \quad (55)$$

and as in F'' terms of higher degree than the second are to be neglected, these equations may be considered accurate for the purpose of the transformation required. Since by equation (33) the eliminant of these equations has the value unity, the discriminant of F'' will be equal to that of F', as has already appeared from the consideration of the principle of conservation of probability of phase, which is, in fact, essentially the same as that expressed by equation (33).

At the time t', the phases satisfying the equation

$$F' = k, \quad (56)$$

where k is any positive constant, have the probability-coefficient Ce^{-k}. At the time t'', the corresponding phases satisfy the equation

$$F'' = k, \quad (57)$$

and have the same probability-coefficient. So also the phases within the limits given by one or the other of these equations are corresponding phases, and have probability-coefficients greater than Ce^{-k}, while phases without these limits have less probability-coefficients. The probability that the phase at the time t' falls within the limits $F' = k$ is the same as the probability that it falls within the limits $F'' = k$ at the time t'', since either event necessitates the other. This probability may be evaluated as follows. We may omit the accents, as we need only consider a single time. Let us denote the extension-in-phase within the limits $F = k$ by U, and the probability that the phase falls within these limits by R, also the extension-in-phase within the limits $F = 1$ by U_1. We have then by definition

$$U = \int^{F=k} \ldots \int dp_1 \ldots dq_n, \quad (58)$$

$$R = \int^{F=k} \cdots \int C\, e^{-F}\, dp_1 \ldots dq_n, \tag{59}$$

$$U_1 = \int^{F=1} \cdots \int dp_1 \ldots dq_n. \tag{60}$$

But since F is a homogeneous quadratic function of the differences

$$p_1 - P_1, p_2 - P_2, \ldots q_n - Q_n,$$

we have identically

$$\begin{aligned} &\int^{F=k} \cdots \int d(p_1 - P_1) \ldots d(q_n - Q_n) \\ &= \int^{kF=k} \cdots \int k^n\, d(p_1 - P_1) \ldots d(q_n - Q_n) \\ &= k^n \int^{F=1} \cdots \int d(p_1 - P_1) \ldots d(q_n - Q_n). \end{aligned}$$

That is
$$U = k^n\, U_1, \tag{61}$$

whence
$$dU = U_1\, n\, k^{n-1}\, dk. \tag{62}$$

But if k varies, equations (58) and (59) give

$$dU = \int_{F=k}^{F=k+dk} \cdots \int dp_1 \ldots dq_n \tag{63}$$

$$dR = \int_{F=k}^{F=k+dk} \cdots \int C\, e^{-F}\, dp_1 \ldots dq_n \tag{64}$$

Since the factor Ce^{-F} has the constant value Ce^{-k} in the last multiple integral, we have

$$dR = C\, e^{-k}\, dU = C\, U_1\, n\, e^{-k}\, k^{n-1}\, dk, \tag{65}$$

$$R = -\, C\, U_1 \underline{|n}\, e^{-k} \left(1 + k + \frac{k^2}{2} + \ldots + \frac{k^{n-1}}{\underline{|n-1}}\right) + \text{const.} \tag{66}$$

We may determine the constant of integration by the condition that R vanishes with k. This gives

$$R = C\,U_1 \lfloor n - C\,U_1 \lfloor n\, e^{-k}\left(1 + k + \frac{k^2}{2} + \ldots + \frac{k^{n-1}}{\lfloor n-1}\right). \quad (67)$$

We may determine the value of the constant U_1 by the condition that $R = 1$ for $k = \infty$. This gives $C\,U_1 \lfloor n = 1$, and

$$R = 1 - e^{-k}\left(1 + k + \frac{k^2}{2} \ldots + \frac{k^{n-1}}{\lfloor n-1}\right), \quad (68)$$

$$U = \frac{k^n}{C \lfloor n}. \quad (69)$$

It is worthy of notice that the form of these equations depends only on the number of degrees of freedom of the system, being in other respects independent of its dynamical nature, except that the forces must be functions of the coördinates either alone or with the time.

If we write

$$k_{R=\frac{1}{2}}$$

for the value of k which substituted in equation (68) will give $R = \frac{1}{2}$, the phases determined by the equation

$$F = k_{R=\frac{1}{2}} \quad (70)$$

will have the following properties.

The probability that the phase falls within the limits formed by these phases is greater than the probability that it falls within any other limits enclosing an equal extension-in-phase. It is equal to the probability that the phase falls without the same limits.

These properties are analogous to those which in the theory of errors in the determination of a single quantity belong to values expressed by $A \pm a$, when A is the most probable value and a the 'probable error.'

CHAPTER III.

APPLICATION OF THE PRINCIPLE OF CONSERVATION OF EXTENSION-IN-PHASE TO THE INTEGRATION OF THE DIFFERENTIAL EQUATIONS OF MOTION.*

We have seen that the principle of conservation of extension-in-phase may be expressed as a differential relation between the coördinates and momenta and the arbitrary constants of the integral equations of motion. Now the integration of the differential equations of motion consists in the determination of these constants as functions of the coördinates and momenta with the time, and the relation afforded by the principle of conservation of extension-in-phase may assist us in this determination.

It will be convenient to have a notation which shall not distinguish between the coördinates and momenta. If we write $r_1 \ldots r_{2n}$ for the coördinates and momenta, and $a \ldots h$ as before for the arbitrary constants, the principle of which we wish to avail ourselves, and which is expressed by equation (37), may be written

$$\frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} = \text{func.}(a, \ldots h). \tag{71}$$

Let us first consider the case in which the forces are determined by the coördinates alone. Whether the forces are 'conservative' or not is immaterial. Since the differential equations of motion do not contain the time (t) in the finite form, if we eliminate dt from these equations, we obtain $2n-1$ equations in $r_1, \ldots r_{2n}$ and their differentials, the integration of which will introduce $2n-1$ arbitrary constants which we shall call $b \ldots h$. If we can effect these integrations, the

* See Boltzmann: "Zusammenhang zwischen den Sätzen über das Verhalten mehratomiger Gasmolecüle mit Jacobi's Princip des letzten Multiplicators. Sitzb. der Wiener Akad., Bd. LXIII, Abth. II., S. 679, (1871).

remaining constant (a) will then be introduced in the final integration, (viz., that of an equation containing dt,) and will be added to or subtracted from t in the integral equation. Let us have it subtracted from t. It is evident then that

$$\frac{dr_1}{da} = -\dot{r}_1, \quad \frac{dr_2}{da} = -\dot{r}_2, \quad \text{etc.} \tag{72}$$

Moreover, since $b, \ldots h$ and $t - a$ are independent functions of $r_1, \ldots r_{2n}$, the latter variables are functions of the former. The Jacobian in (71) is therefore function of $b, \ldots h$, and $t - a$, and since it does not vary with t it cannot vary with a. We have therefore in the case considered, viz., where the forces are functions of the coördinates alone,

$$\frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} = \text{func.}(b, \ldots h). \tag{73}$$

Now let us suppose that of the first $2n - 1$ integrations we have accomplished all but one, determining $2n - 2$ arbitrary constants (say $c, \ldots h$) as functions of $r_1, \ldots r_{2n}$, leaving b as well as a to be determined. Our $2n - 2$ finite equations enable us to regard all the variables $r_1, \ldots r_{2n}$, and all functions of these variables as functions of two of them, (say r_1 and r_2,) with the arbitrary constants $c, \ldots h$. To determine b, we have the following equations for constant values of $c, \ldots h$.

$$dr_1 = \frac{dr_1}{da}\,da + \frac{dr_1}{db}\,db,$$

$$dr_2 = \frac{dr_2}{da}\,da + \frac{dr_2}{db}\,db,$$

whence

$$\frac{d(r_1, r_2)}{d(a, b)}\,db = -\frac{dr_2}{da}\,dr_1 + \frac{dr_1}{da}\,dr_2. \tag{74}$$

Now, by the ordinary formula for the change of variables,

$$\int\cdots\int \frac{d(r_1, r_2)}{d(a, b)}\,da\,db\,dr_3 \ldots dr_{2n} = \int\cdots\int dr_1 \ldots dr_{2n}$$

$$= \int\cdots\int \frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)}\,da \ldots dh$$

$$= \int\cdots\int \frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} \frac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})}\,da\,db\,dr_3 \ldots dr_{2n},$$

where the limits of the multiple integrals are formed by the same phases. Hence

$$\frac{d(r_1, r_2)}{d(a, b)} = \frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} \frac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})}, \tag{75}$$

With the aid of this equation, which is an identity, and (72), we may write equation (74) in the form

$$\frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} \frac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})} db = \dot{r}_2 \, dr_1 - \dot{r}_1 \, dr_2. \tag{76}$$

The separation of the variables is now easy. The differential equations of motion give $\dot{r}_1$ and $\dot{r}_2$ in terms of $r_1, \ldots r_{2n}$. The integral equations already obtained give $c, \ldots h$ and therefore the Jacobian $d(c, \ldots h)/d(r_3, \ldots r_{2n})$, in terms of the same variables. But in virtue of these same integral equations, we may regard functions of $r_1, \ldots r_{2n}$ as functions of r_1 and r_2 with the constants $c, \ldots h$. If therefore we write the equation in the form

$$\frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} db = \frac{\dot{r}_2}{\dfrac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})}} dr_1 - \frac{\dot{r}_1}{\dfrac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})}} dr_2, \tag{77}$$

the coefficients of dr_1 and dr_2 may be regarded as known functions of r_1 and r_2 with the constants $c, \ldots h$. The coefficient of db is by (73) a function of $b, \ldots h$. It is not indeed a known function of these quantities, but since $c, \ldots h$ are regarded as constant in the equation, we know that the first member must represent the differential of some function of $b, \ldots h$, for which we may write b'. We have thus

$$db' = \frac{\dot{r}_2}{\dfrac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})}} dr_1 - \frac{\dot{r}_1}{\dfrac{d(c_1, \ldots h)}{d(r_3, \ldots r_{2n})}} dr_2, \tag{78}$$

which may be integrated by quadratures and gives b' as functions of $r_1, r_2, \ldots c, \ldots h$, and thus as function of $r_1, \ldots r_{2n}$.

This integration gives us the last of the arbitrary constants which are functions of the coördinates and momenta without the time. The final integration, which introduces the remain-

ing constant (a), is also a quadrature, since the equation to be integrated may be expressed in the form

$$dt = F(r_1)\, dr_1.$$

Now, apart from any such considerations as have been adduced, if we limit ourselves to the changes which take place in time, we have identically

$$\dot{r}_2\, dr_1 - \dot{r}_1\, dr_2 = 0,$$

and $\dot{r}_1$ and $\dot{r}_2$ are given in terms of $r_1, \ldots r_{2n}$ by the differential equations of motion. When we have obtained $2n - 2$ integral equations, we may regard $\dot{r}_2$ and $\dot{r}_1$ as known functions of r_1 and r_2. The only remaining difficulty is in integrating this equation. If the case is so simple as to present no difficulty, or if we have the skill or the good fortune to perceive that the multiplier

$$\frac{1}{\dfrac{d(c, \ldots h)}{d(r_3, \ldots r_{2n})}}, \tag{79}$$

or any other, will make the first member of the equation an exact differential, we have no need of the rather lengthy considerations which have been adduced. The utility of the principle of conservation of extension-in-phase is that it supplies a 'multiplier' which renders the equation integrable, and which it might be difficult or impossible to find otherwise.

It will be observed that the function represented by b' is a particular case of that represented by b. The system of arbitrary constants a, b', $c \ldots h$ has certain properties notable for simplicity. If we write b' for b in (77), and compare the result with (78), we get

$$\frac{d(r_1 \ldots r_{2n})}{d(a, b', c, \ldots h)} = 1. \tag{80}$$

Therefore the multiple integral

$$\int \ldots \int da\, db'\, dc \ldots dh \tag{81}$$

taken within limits formed by phases regarded as contemporaneous represents the extension-in-phase within those limits.

The case is somewhat different when the forces are not determined by the coördinates alone, but are functions of the coördinates with the time. All the arbitrary constants of the integral equations must then be regarded in the general case as functions of $r_1, \ldots r_{2n}$, and t. We cannot use the principle of conservation of extension-in-phase until we have made $2n-1$ integrations. Let us suppose that the constants $b, \ldots h$ have been determined by integration in terms of $r_1, \ldots r_{2n}$, and t, leaving a single constant (a) to be thus determined. Our $2n-1$ finite equations enable us to regard all the variables $r_1, \ldots r_{2n}$ as functions of a single one, say r_1.

For constant values of $b, \ldots h$, we have

$$dr_1 = \frac{dr_1}{da}\, da + \dot{r}_1\, dt. \tag{82}$$

Now

$$\int \cdots \int \frac{dr_1}{da}\, da\, dr_2 \ldots dr_{2n} = \int \cdots \int dr_1 \ldots d r_{2n}$$

$$= \int \cdots \int \frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)}\, da \ldots dh$$

$$= \int \cdots \int \frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} \frac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}\, da\, dr_2 \ldots dr_{2n},$$

where the limits of the integrals are formed by the same phases. We have therefore

$$\frac{dr_1}{da} = \frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)} \frac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}, \tag{83}$$

by which equation (82) may be reduced to the form

$$\frac{d(r_1, \ldots r_{2n})}{d(a, \ldots h)}\, da = \frac{1}{\dfrac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}}\, dr_1 - \frac{\dot{r}_1}{\dfrac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}}\, dt. \tag{84}$$

Now we know by (71) that the coefficient of da is a function of $a, \ldots h$. Therefore, as $b, \ldots h$ are regarded as constant in the equation, the first number represents the differential

of a function of $a, \ldots h$, which we may denote by a'. We have then

$$da' = \frac{1}{\frac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}} dr_1 - \frac{\dot{r}_1'}{\frac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}} dt, \tag{85}$$

which may be integrated by quadratures. In this case we may say that the principle of conservation of extension-in-phase has supplied the 'multiplier'

$$\frac{1}{\frac{d(b, \ldots h)}{d(r_2, \ldots r_{2n})}} \tag{86}$$

for the integration of the equation

$$dr_1 - \dot{r}_1 \, dt = \dot{0}. \tag{87}$$

The system of arbitrary constants $a', b, \ldots h$ has evidently the same properties which were noticed in regard to the system $a, b', \ldots h$.

CHAPTER IV.

ON THE DISTRIBUTION IN PHASE CALLED CANONICAL, IN WHICH THE INDEX OF PROBABILITY IS A LINEAR FUNCTION OF THE ENERGY.

LET us now give our attention to the statistical equilibrium of ensembles of conservation systems, especially to those cases and properties which promise to throw light on the phenomena of thermodynamics.

The condition of statistical equilibrium may be expressed in the form*

$$\Sigma\left(\frac{dP}{dp_1}\dot{p}_1 + \frac{dP}{dq_1}\dot{q}_1\right) = 0, \tag{88}$$

where P is the coefficient of probability, or the quotient of the density-in-phase by the whole number of systems. To satisfy this condition, it is necessary and sufficient that P should be a function of the p's and q's (the momenta and coördinates) which does not vary with the time in a moving system. In all cases which we are now considering, the energy, or any function of the energy, is such a function.

$$P = \text{func. } (\epsilon)$$

will therefore satisfy the equation, as indeed appears identically if we write it in the form

$$\Sigma\left(\frac{dP}{dq_1}\frac{d\epsilon}{dp_1} - \frac{dP}{dp_1}\frac{d\epsilon}{dq_1}\right) = 0.$$

There are, however, other conditions to which P is subject, which are not so much conditions of statistical equilibrium, as conditions implicitly involved in the definition of the coeffi-

* See equations (20), (41), (42), also the paragraph following equation (20). The positions of any external bodies which can affect the systems are here supposed uniform for all the systems and constant in time.

cient of probability, whether the case is one of equilibrium or not. These are: that P should be single-valued, and neither negative nor imaginary for any phase, and that expressed by equation (46), viz.,

$$\int_{\text{phases}}^{\text{all}} \cdots \int P\, dp_1 \ldots dq_n = 1. \tag{89}$$

These considerations exclude

$$P = \epsilon \times \text{constant},$$

as well as

$$P = \text{constant},$$

as cases to be considered.

The distribution represented by

$$\eta = \log P = \frac{\psi - \epsilon}{\Theta}, \tag{90}$$

or

$$P = e^{\frac{\psi - \epsilon}{\Theta}}, \tag{91}$$

where Θ and ψ are constants, and Θ positive, seems to represent the most simple case conceivable, since it has the property that when the system consists of parts with separate energies, the laws of the distribution in phase of the separate parts are of the same nature,— a property which enormously simplifies the discussion, and is the foundation of extremely important relations to thermodynamics. The case is not rendered less simple by the divisor Θ, (a quantity of the same dimensions as ϵ,) but the reverse, since it makes the distribution independent of the units employed. The negative sign of ϵ is required by (89), which determines also the value of ψ for any given Θ, viz.,

$$e^{-\frac{\psi}{\Theta}} = \int_{\text{phases}}^{\text{all}} \cdots \int e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n. \tag{92}$$

When an ensemble of systems is distributed in phase in the manner described, *i. e.*, when the index of probability is a

linear function of the energy, we shall say that the ensemble is *canonically distributed*, and shall call the divisor of the energy (Θ) the *modulus* of distribution.

The fractional part of an ensemble canonically distributed which lies within any given limits of phase is therefore represented by the multiple integral

$$\int \dots \int e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \dots dq_n \qquad (93)$$

taken within those limits. We may express the same thing by saying that the multiple integral expresses the probability that an unspecified system of the ensemble (*i. e.*, one of which we only know that it belongs to the ensemble) falls within the given limits.

Since the value of a multiple integral of the form (23) (which we have called an extension-in-phase) bounded by any given phases is independent of the system of coördinates by which it is evaluated, the same must be true of the multiple integral in (92), as appears at once if we divide up this integral into parts so small that the exponential factor may be regarded as constant in each. The value of ψ is therefore independent of the system of coördinates employed.

It is evident that ψ might be defined as the energy for which the coefficient of probability of phase has the value unity. Since however this coefficient has the dimensions of the inverse nth power of the product of energy and time,* the energy represented by ψ is not independent of the units of energy and time. But when these units have been chosen, the definition of ψ will involve the same arbitrary constant as ϵ, so that, while in any given case the numerical values of ψ or ϵ will be entirely indefinite until the zero of energy has also been fixed for the system considered, the difference $\psi - \epsilon$ will represent a perfectly definite amount of energy, which is entirely independent of the zero of energy which we may choose to adopt.

* See Chapter I, p. 19.

It is evident that the canonical distribution is entirely determined by the modulus (considered as a quantity of energy) and the nature of the system considered, since when equation (92) is satisfied the value of the multiple integral (93) is independent of the units and of the coördinates employed, and of the zero chosen for the energy of the system.

In treating of the canonical distribution, we shall always suppose the multiple integral in equation (92) to have a finite value, as otherwise the coefficient of probability vanishes, and the law of distribution becomes illusory. This will exclude certain cases, but not such apparently, as will affect the value of our results with respect to their bearing on thermodynamics. It will exclude, for instance, cases in which the system or parts of it can be distributed in unlimited space (or in a space which has limits, but is still infinite in volume), while the energy remains beneath a finite limit. It also excludes many cases in which the energy can decrease without limit, as when the system contains material points which attract one another inversely as the squares of their distances. Cases of material points attracting each other inversely as the distances would be excluded for some values of Θ, and not for others. The investigation of such points is best left to the particular cases. For the purposes of a general discussion, it is sufficient to call attention to the assumption implicitly involved in the formula (92).*

The modulus Θ has properties analogous to those of temperature in thermodynamics. Let the system A be defined as one of an ensemble of systems of m degrees of freedom distributed in phase with a probability-coefficient

$$e^{\frac{\psi_A - \epsilon_A}{\Theta}},$$

* It will be observed that similar limitations exist in thermodynamics. In order that a mass of gas can be in thermodynamic equilibrium, it is necessary that it be enclosed. There is no thermodynamic equilibrium of a (finite) mass of gas in an infinite space. Again, that two attracting particles should be able to do an infinite amount of work in passing from one configuration (which is regarded as possible) to another, is a notion which, although perfectly intelligible in a mathematical formula, is quite foreign to our ordinary conceptions of matter.

and the system B as one of an ensemble of systems of n degrees of freedom distributed in phase with a probability-coefficient

$$e^{\frac{\psi_B - \epsilon_B}{\Theta}},$$

which has the same modulus. Let $q_1, \ldots q_m, p_1, \ldots p_m$ be the coördinates and momenta of A, and $q_{m+1}, \ldots q_{m+n}, p_{m+1}, \ldots p_{m+n}$ those of B. Now we may regard the systems A and B as together forming a system C, having $m + n$ degrees of freedom, and the coördinates and momenta $q_1, \ldots q_{m+n}, p_1, \ldots p_{m+n}$. The probability that the phase of the system C, as thus defined, will fall within the limits

$$dp_1, \ldots dp_{m+n}, dq_1, \ldots dq_{m+n}$$

is evidently the product of the probabilities that the systems A and B will each fall within the specified limits, viz.,

$$e^{\frac{\psi_A + \psi_B - \epsilon_A - \epsilon_B}{\Theta}} dp_1 \ldots dp_{m+n} dq_1 \ldots dq_{m+n}. \tag{94}$$

We may therefore regard C as an undetermined system of an ensemble distributed with the probability-coefficient

$$e^{\frac{\psi_A + \psi_B - (\epsilon_A + \epsilon_B)}{\Theta}}, \tag{95}$$

an ensemble which might be defined as formed by combining each system of the first ensemble with each of the second. But since $\epsilon_A + \epsilon_B$ is the energy of the whole system, and ψ_A and ψ_B are constants, the probability-coefficient is of the general form which we are considering, and the ensemble to which it relates is in statistical equilibrium and is canonically distributed.

This result, however, so far as statistical equilibrium is concerned, is rather nugatory, since conceiving of separate systems as forming a single system does not create any interaction between them, and if the systems combined belong to ensembles in statistical equilibrium, to say that the ensemble formed by such combinations as we have supposed is in statistical equilibrium, is only to repeat the data in different

words. Let us therefore suppose that in forming the system C we add certain forces acting between A and B, and having the force-function $-\epsilon_{AB}$. The energy of the system C is now $\epsilon_A + \epsilon_B + \epsilon_{AB}$, and an ensemble of such systems distributed with a density proportional to

$$e^{\frac{-(\epsilon_A+\epsilon_B+\epsilon_{AB})}{\Theta}} \tag{96}$$

would be in statistical equilibrium. Comparing this with the probability-coefficient of C given above (95), we see that if we suppose ϵ_{AB} (or rather the variable part of this term when we consider all possible configurations of the systems A and B) to be infinitely small, the actual distribution in phase of C will differ infinitely little from one of statistical equilibrium, which is equivalent to saying that its distribution in phase will vary infinitely little even in a time indefinitely prolonged.* The case would be entirely different if A and B belonged to ensembles having different moduli, say Θ_A and Θ_B. The probability-coefficient of C would then be

$$e^{\frac{\psi_A-\epsilon_A}{\Theta_A}+\frac{\psi_B-\epsilon_B}{\Theta_B}}, \tag{97}$$

which is not approximately proportional to any expression of the form (96).

Before proceeding farther in the investigation of the distribution in phase which we have called canonical, it will be interesting to see whether the properties with respect to

* It will be observed that the above condition relating to the forces which act between the different systems is entirely analogous to that which must hold in the corresponding case in thermodynamics. The most simple test of the equality of temperature of two bodies is that they remain in equilibrium when brought into thermal contact. Direct thermal contact implies molecular forces acting between the bodies. Now the test will fail unless the energy of these forces can be neglected in comparison with the other energies of the bodies. Thus, in the case of energetic chemical action between the bodies, or when the number of particles affected by the forces acting between the bodies is not negligible in comparison with the whole number of particles (as when the bodies have the form of exceedingly thin sheets), the contact of bodies of the same temperature may produce considerable thermal disturbance, and thus fail to afford a reliable criterion of the equality of temperature.

statistical equilibrium which have been described are peculiar to it, or whether other distributions may have analogous properties.

Let η' and η'' be the indices of probability in two independent ensembles which are each in statistical equilibrium, then $\eta' + \eta''$ will be the index in the ensemble obtained by combining each system of the first ensemble with each system of the second. This third ensemble will of course be in statistical equilibrium, and the function of phase $\eta' + \eta''$ will be a constant of motion. Now when infinitesimal forces are added to the compound systems, if $\eta' + \eta''$ or a function differing infinitesimally from this is still a constant of motion, it must be on account of the nature of the forces added, or if their action is not entirely specified, on account of conditions to which they are subject. Thus, in the case already considered, $\eta' + \eta''$ is a function of the energy of the compound system, and the infinitesimal forces added are subject to the law of conservation of energy.

Another natural supposition in regard to the added forces is that they should be such as not to affect the moments of momentum of the compound system. To get a case in which moments of momentum of the compound system shall be constants of motion, we may imagine material particles contained in two concentric spherical shells, being prevented from passing the surfaces bounding the shells by repulsions acting always in lines passing through the common centre of the shells. Then, if there are no forces acting between particles in different shells, the mass of particles in each shell will have, besides its energy, the moments of momentum about three axes through the centre as constants of motion.

Now let us imagine an ensemble formed by distributing in phase the system of particles in one shell according to the index of probability

$$A - \frac{\epsilon}{\Theta} + \frac{\omega_1}{\Omega_1} + \frac{\omega_2}{\Omega_2} + \frac{\omega_3}{\Omega_3}, \tag{98}$$

where ϵ denotes the energy of the system, and ω_1, ω_2, ω_3, its three moments of momentum, and the other letters constants.

In like manner let us imagine a second ensemble formed by distributing in phase the system of particles in the other shell according to the index

$$A' - \frac{\epsilon'}{\Theta} + \frac{\omega_1'}{\Omega_1} + \frac{\omega_2'}{\Omega_2} + \frac{\omega_3'}{\Omega_3}, \tag{99}$$

where the letters have similar significations, and $\Theta, \Omega_1, \Omega_2, \Omega_3$ the same values as in the preceding formula. Each of the two ensembles will evidently be in statistical equilibrium, and therefore also the ensemble of compound systems obtained by combining each system of the first ensemble with each of the second. In this third ensemble the index of probability will be

$$A + A' - \frac{\epsilon + \epsilon'}{\Theta} + \frac{\omega_1 + \omega_1'}{\Omega_1} + \frac{\omega_2 + \omega_2'}{\Omega_2} + \frac{\omega_3 + \omega_3'}{\Omega_3}, \tag{100}$$

where the four numerators represent functions of phase which are constants of motion for the compound systems.

Now if we add in each system of this third ensemble infinitesimal conservative forces of attraction or repulsion between particles in different shells, determined by the same law for all the systems, the functions $\omega_1 + \omega'$, $\omega_2 + \omega_2'$, and $\omega_3 + \omega_3'$ will remain constants of motion, and a function differing infinitely little from $\epsilon_1 + \epsilon'$ will be a constant of motion. It would therefore require only an infinitesimal change in the distribution in phase of the ensemble of compound systems to make it a case of statistical equilibrium. These properties are entirely analogous to those of canonical ensembles.*

Again, if the relations between the forces and the coördinates can be expressed by linear equations, there will be certain "normal" types of vibration of which the actual motion may be regarded as composed, and the whole energy may be divided

* It would not be possible to omit the term relating to energy in the above indices, since without this term the condition expressed by equation (89) cannot be satisfied.

The consideration of the above case of statistical equilibrium may be made the foundation of the theory of the thermodynamic equilibrium of rotating bodies, — a subject which has been treated by Maxwell in his memoir "On Boltzmann's theorem on the average distribution of energy in a system of material points." Cambr. Phil. Trans., vol. XII, p. 547, (1878).

into parts relating separately to vibrations of these different types. These partial energies will be constants of motion, and if such a system is distributed according to an index which is any function of the partial energies, the ensemble will be in statistical equilibrium. Let the index be a linear function of the partial energies, say

$$A - \frac{\epsilon_1}{\Theta_1} \cdots - \frac{\epsilon_n}{\Theta_n}. \tag{101}$$

Let us suppose that we have also a second ensemble composed of systems in which the forces are linear functions of the coördinates, and distributed in phase according to an index which is a linear function of the partial energies relating to the normal types of vibration, say

$$A' - \frac{\epsilon_1'}{\Theta_1'} \cdots - \frac{\epsilon_m'}{\Theta_m'}. \tag{102}$$

Since the two ensembles are both in statistical equilibrium, the ensemble formed by combining each system of the first with each system of the second will also be in statistical equilibrium. Its distribution in phase will be represented by the index

$$A + A' - \frac{\epsilon_1}{\Theta_1} \cdots - \frac{\epsilon_n}{\Theta_n} - \frac{\epsilon_1'}{\Theta_1'} \cdots - \frac{\epsilon_m'}{\Theta_m'}, \tag{103}$$

and the partial energies represented by the numerators in the formula will be constants of motion of the compound systems which form this third ensemble.

Now if we add to these compound systems infinitesimal forces acting between the component systems and subject to the same general law as those already existing, viz., that they are conservative and linear functions of the coördinates, there will still be $n + m$ types of normal vibration, and $n + m$ partial energies which are independent constants of motion. If all the original $n + m$ normal types of vibration have different periods, the new types of normal vibration will differ infinitesimally from the old, and the new partial energies, which are constants of motion, will be nearly the same functions of phase as the old. Therefore the distribution in phase of the

ensemble of compound systems after the addition of the supposed infinitesimal forces will differ infinitesimally from one which would be in statistical equilibrium.

The case is not so simple when some of the normal types of motion have the same periods. In this case the addition of infinitesimal forces may completely change the normal types of motion. But the sum of the partial energies for all the original types of vibration which have any same period, will be nearly identical (as a function of phase, *i. e.*, of the coördinates and momenta,) with the sum of the partial energies for the normal types of vibration which have the same, or nearly the same, period after the addition of the new forces. If, therefore, the partial energies in the indices of the first two ensembles (101) and (102) which relate to types of vibration having the same periods, have the same divisors, the same will be true of the index (103) of the ensemble of compound systems, and the distribution represented will differ infinitesimally from one which would be in statistical equilibrium after the addition of the new forces.*

The same would be true if in the indices of each of the original ensembles we should substitute for the term or terms relating to any period which does not occur in the other ensemble, any function of the total energy related to that period, subject only to the general limitation expressed by equation (89). But in order that the ensemble of compound systems (with the added forces) shall always be approximately in statistical equilibrium, it is necessary that the indices of the original ensembles should be linear functions of those partial energies which relate to vibrations of periods common to the two ensembles, and that the coefficients of such partial energies should be the same in the two indices.†

* It is interesting to compare the above relations with the laws respecting the exchange of energy between bodies by radiation, although the phenomena of radiations lie entirely without the scope of the present treatise, in which the discussion is limited to systems of a finite number of degrees of freedom.

† The above may perhaps be sufficiently illustrated by the simple case where $n = 1$ in each system. If the periods are different in the two systems, they may be distributed according to any functions of the energies: but if

The properties of canonically distributed ensembles of systems with respect to the equilibrium of the new ensembles which may be formed by combining each system of one ensemble with each system of another, are therefore not peculiar to them in the sense that analogous properties do not belong to some other distributions under special limitations in regard to the systems and forces considered. Yet the canonical distribution evidently constitutes the most simple case of the kind, and that for which the relations described hold with the least restrictions.

Returning to the case of the canonical distribution, we shall find other analogies with thermodynamic systems, if we suppose, as in the preceding chapters,* that the potential energy (ϵ_q) depends not only upon the coordinates $q_1 \ldots q_n$ which determine the configuration of the system, but also upon certain coördinates a_1, a_2, etc. of bodies which we call *external*, meaning by this simply that they are not to be regarded as forming any part of the system, although their positions affect the forces which act on the system. The forces exerted by the system upon these external bodies will be represented by $-d\epsilon_q/da_1$, $-d\epsilon_q/da_2$, etc., while $-d\epsilon_q/dq_1$, $\ldots -d\epsilon_q/dq_n$ represent all the forces acting upon the bodies of the system, including those which depend upon the position of the external bodies, as well as those which depend only upon the configuration of the system itself. It will be understood that ϵ_p depends only upon $q_1, \ldots q_n, p_1, \ldots p_n$, in other words, that the kinetic energy of the bodies which we call external forms no part of the kinetic energy of the system. It follows that we may write

$$\frac{d\epsilon}{da_1} = \frac{d\epsilon_q}{da_1} = -A_1, \tag{104}$$

although a similar equation would not hold for differentiations relative to the internal coördinates.

the periods are the same they must be distributed canonically with same modulus in order that the compound ensemble with additional forces may be in statistical equilibrium.

* See especially Chapter I, p. 4.

We always suppose these external coördinates to have the same values for all systems of any ensemble. In the case of a canonical distribution, *i. e.*, when the index of probability of phase is a linear function of the energy, it is evident that the values of the external coördinates will affect the distribution, since they affect the energy. In the equation

$$e^{-\frac{\psi}{\Theta}} = \int_{\text{phases}}^{\text{all}} \cdots \int e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n, \tag{105}$$

by which ψ may be determined, the external coördinates, a_1, a_2, etc., contained implicitly in ϵ, as well as Θ, are to be regarded as constant in the integrations indicated. The equation indicates that ψ is a function of these constants. If we imagine their values varied, and the ensemble distributed canonically according to their new values, we have by differentiation of the equation

$$\begin{aligned} e^{-\frac{\psi}{\Theta}}\left(-\frac{1}{\Theta}\,d\psi + \frac{\psi}{\Theta^2}\,d\Theta\right) &= \frac{1}{\Theta^2}\,d\Theta \int_{\text{phases}}^{\text{all}} \cdots \int \epsilon\, e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n \\ &\quad - \frac{1}{\Theta}\,da_1 \int_{\text{phases}}^{\text{all}} \cdots \int \frac{d\epsilon}{da_1}\, e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n \\ &\quad - \frac{1}{\Theta}\,da_2 \int_{\text{phases}}^{\text{all}} \cdots \int \frac{d\epsilon}{da_2}\, e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n - \text{etc.}, \end{aligned} \tag{106}$$

or, multiplying by $\Theta\, e^{\frac{\psi}{\Theta}}$, and setting

$$-\frac{d\epsilon}{da_1} = A_1, \qquad -\frac{d\epsilon}{da_2} = A_2, \quad \text{etc.},$$

$$\begin{aligned} -d\psi + \frac{\psi}{\Theta}\,d\Theta &= \frac{1}{\Theta}\,d\Theta \int_{\text{phases}}^{\text{all}} \cdots \int \epsilon\, e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n \\ &\quad + da_1 \int_{\text{phases}}^{\text{all}} \cdots \int A_1\, e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n \\ &\quad + da_2 \int_{\text{phases}}^{\text{all}} \cdots \int A_2\, e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n + \text{etc.} \end{aligned} \tag{107}$$

Now the average value in the ensemble of any quantity (which we shall denote in general by a horizontal line above the proper symbol) is determined by the equation

$$\bar{u} = \int_{\text{phases}}^{\text{all}} \cdots \int u\, e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \ldots dq_n. \tag{108}$$

Comparing this with the preceding equation, we have

$$d\psi = \frac{\bar{\psi}}{\Theta}\, d\Theta - \frac{\bar{\epsilon}}{\Theta}\, d\Theta - \bar{A}_1\, da_1 - \bar{A}_2\, da_2 - \text{etc.} \tag{109}$$

Or, since

$$\frac{\psi - \epsilon}{\Theta} = \eta, \tag{110}$$

and

$$\frac{\psi - \bar{\epsilon}}{\Theta} = \bar{\eta}, \tag{111}$$

$$d\psi = \bar{\eta}\, d\Theta - \bar{A}_1\, da_1 - \bar{A}_2\, da_2 - \text{etc.} \tag{112}$$

Moreover, since (111) gives

$$d\psi - d\bar{\epsilon} = \Theta\, d\bar{\eta} + \bar{\eta}\, d\Theta, \tag{113}$$

we have also

$$d\bar{\epsilon} = -\Theta\, d\bar{\eta} - \bar{A}_1\, da_1 - \bar{A}_2\, da_2 - \text{etc.} \tag{114}$$

This equation, if we neglect the sign of averages, is identical in form with the thermodynamic equation

$$d\eta = \frac{d\epsilon + A_1\, da_1 + A_2\, da_2 + \text{etc.}}{T}, \tag{115}$$

or

$$d\epsilon = T\, d\eta - A_1\, da_1 - A_2\, da_2 - \text{etc.}, \tag{116}$$

which expresses the relation between the energy, temperature, and entropy of a body in thermodynamic equilibrium, and the forces which it exerts on external bodies, — a relation which is the mathematical expression of the second law of thermodynamics for reversible changes. The modulus in the statistical equation corresponds to temperature in the thermodynamic equation, and the average index of probability *with its sign reversed* corresponds to entropy. But in the thermodynamic equation the entropy (η) is a quantity which is

only defined by the equation itself, and incompletely defined in that the equation only determines its differential, and the constant of integration is arbitrary. On the other hand, the $\bar{\eta}$ in the statistical equation has been completely defined as the average value in a canonical ensemble of systems of the logarithm of the coefficient of probability of phase.

We may also compare equation (112) with the thermodynamic equation

$$\psi = -\eta dT - A_1 da_1 - A_2 da_2 - \text{etc.}, \qquad (117)$$

where ψ represents the function obtained by subtracting the product of the temperature and entropy from the energy.

How far, or in what sense, the similarity of these equations constitutes any demonstration of the thermodynamic equations, or accounts for the behavior of material systems, as described in the theorems of thermodynamics, is a question of which we shall postpone the consideration until we have further investigated the properties of an ensemble of systems distributed in phase according to the law which we are considering. The analogies which have been pointed out will at least supply the motive for this investigation, which will naturally commence with the determination of the average values in the ensemble of the most important quantities relating to the systems, and to the distribution of the ensemble with respect to the different values of these quantities.

CHAPTER V.

AVERAGE VALUES IN A CANONICAL ENSEMBLE OF SYSTEMS.

In the simple but important case of a system of material points, if we use rectangular coördinates, we have for the product of the differentials of the coördinates

$$dx_1\, dy_1\, dz_1 \ldots dx_\nu\, dy_\nu\, dz_\nu,$$

and for the product of the differentials of the momenta

$$m_1\, d\dot{x}_1\, m_1\, d\dot{y}_1\, m_1\, d\dot{z}_1 \ldots m_\nu\, d\dot{x}_\nu\, m_\nu\, d\dot{y}_\nu\, m_\nu\, d\dot{z}_\nu.$$

The product of these expressions, which represents an element of extension-in-phase, may be briefly written

$$m_1\, d\dot{x}_1 \ldots m_\nu\, d\dot{z}_\nu\, dx_1 \ldots dz_\nu;$$

and the integral

$$\int \ldots \int e^{\frac{\psi-\epsilon}{\Theta}} m_1\, d\dot{x}_1 \ldots m_\nu\, d\dot{z}_\nu\, dx_1 \ldots dz_\nu \qquad (118)$$

will represent the probability that a system taken at random from an ensemble canonically distributed will fall within any given limits of phase.

In this case

$$\epsilon = \epsilon_q + \tfrac{1}{2} m_1 \dot{x}_1^2 \ldots + \tfrac{1}{2} m_\nu \dot{z}_\nu^2, \qquad (119)$$

and

$$e^{\frac{\psi-\epsilon}{\Theta}} = e^{\frac{\psi-\epsilon_q}{\Theta}} e^{-\frac{m_1\dot{x}_1^2}{2\Theta}} \ldots e^{-\frac{m_\nu\dot{z}_\nu^2}{2\Theta}}. \qquad (120)$$

The potential energy (ϵ_q) is independent of the velocities, and if the limits of integration for the coördinates are independent of the velocities, and the limits of the several velocities are independent of each other as well as of the coördinates,

the multiple integral may be resolved into the product of integrals

$$\int \cdots \int e^{\frac{\psi-\epsilon_q}{\Theta}} dx_1 \ldots dz_\nu \int e^{-\frac{m_1\dot{x}_1^2}{2\Theta}} m_1 d\dot{x}_1 \ldots \int e^{-\frac{m_\nu \dot{z}_\nu^2}{2\Theta}} m_\nu d\dot{z}_\nu. \quad (121)$$

This shows that the probability that the configuration lies within any given limits is independent of the velocities, and that the probability that any component velocity lies within any given limits is independent of the other component velocities and of the configuration.

Since

$$\int_{-\infty}^{+\infty} e^{-\frac{m_1\dot{x}_1^2}{2\Theta}} m_1 \, d\dot{x}_1 = \sqrt{2\pi m_1 \Theta}, \quad (122)$$

and

$$\int_{-\infty}^{+\infty} \tfrac{1}{2} m_1 \dot{x}_1^2 e^{-\frac{m_1\dot{x}_1^2}{2\Theta}} m_1 \, d\dot{x}_1 = \sqrt{\tfrac{1}{2}\pi m_1 \Theta^3}, \quad (123)$$

the average value of the part of the kinetic energy due to the velocity $\dot{x}_1$, which is expressed by the quotient of these integrals, is $\frac{1}{2}\Theta$. This is true whether the average is taken for the whole ensemble or for any particular configuration, whether it is taken without reference to the other component velocities, or only those systems are considered in which the other component velocities have particular values or lie within specified limits.

The number of coördinates is 3ν or n. We have, therefore, for the average value of the kinetic energy of a system

$$\bar{\epsilon}_p = \tfrac{3}{2}\nu\Theta = \tfrac{1}{2} n \Theta. \quad (124)$$

This is equally true whether we take the average for the whole ensemble, or limit the average to a single configuration.

The distribution of the systems with respect to their component velocities follows the 'law of errors'; the probability that the value of any component velocity lies within any given limits being represented by the value of the corresponding integral in (121) for those limits, divided by $(2\pi m \Theta)^{\frac{1}{2}}$,

which is the value of the same integral for infinite limits. Thus the probability that the value of x_1 lies between any given limits is expressed by

$$\left(\frac{m_1}{2\pi\Theta}\right)^{\frac{1}{2}}\int e^{-\frac{m_1\dot{x}_1^2}{2\Theta}}d\dot{x}_1. \tag{125}$$

The expression becomes more simple when the velocity is expressed with reference to the energy involved. If we set

$$s = \left(\frac{m_1}{2\Theta}\right)^{\frac{1}{2}}\dot{x}_1,$$

the probability that s lies between any given limits is expressed by

$$\frac{1}{\sqrt{\pi}}\int e^{-s^2}ds. \tag{126}$$

Here s is the ratio of the component velocity to that which would give the energy Θ; in other words, s^2 is the quotient of the energy due to the component velocity divided by Θ. The distribution with respect to the partial energies due to the component velocities is therefore the same for all the component velocities.

The probability that the configuration lies within any given limits is expressed by the value of

$$M^{\frac{1}{2}}(2\pi\Theta)^{\frac{3\nu}{2}}\int\cdots\int e^{\frac{\psi-\epsilon_q}{\Theta}}dx_1\ldots dz_\nu \tag{127}$$

for those limits, where M denotes the product of all the masses. This is derived from (121) by substitution of the values of the integrals relating to velocities taken for infinite limits.

Very similar results may be obtained in the general case of a conservative system of n degrees of freedom. Since ϵ_p is a homogeneous quadratic function of the p's, it may be divided into parts by the formula

$$\epsilon_p = \tfrac{1}{2}p_1\frac{d\epsilon_p}{dp_1}+\ldots+\tfrac{1}{2}p_n\frac{d\epsilon_p}{dp_n} \tag{128}$$

where ϵ might be written for ϵ_p in the differential coefficients without affecting the signification. The average value of the first of these parts, for any given configuration, is expressed by the quotient

$$\frac{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \tfrac{1}{2} p_1 \frac{d\epsilon}{dp_1} e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dp_n}{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dp_n} \tag{129}$$

Now we have by integration by parts

$$\int_{-\infty}^{+\infty} p_1 e^{\frac{\psi-\epsilon}{\Theta}} \frac{d\epsilon}{dp_1} dp_1 = \Theta \int_{-\infty}^{+\infty} e^{\frac{\psi-\epsilon}{\Theta}} dp_1. \tag{130}$$

By substitution of this value, the above quotient reduces to $\frac{\Theta}{2}$, which is therefore the average value of $\frac{1}{2} p_1 \frac{d\epsilon}{dp_1}$ for the given configuration. Since this value is independent of the configuration, it must also be the average for the whole ensemble, as might easily be proved directly. (To make the preceding proof apply directly to the whole ensemble, we have only to write $dp_1 \ldots dq_n$ for $dp_1 \ldots dp_n$ in the multiple integrals.) This gives $\frac{1}{2} n \Theta$ for the average value of the whole kinetic energy for any given configuration, or for the whole ensemble, as has already been proved in the case of material points.

The mechanical significance of the several parts into which the kinetic energy is divided in equation (128) will be apparent if we imagine that by the application of suitable forces (different from those derived from ϵ_q and so much greater that the latter may be neglected in comparison) the system was brought from rest to the state of motion considered, so rapidly that the configuration was not sensibly altered during the process, and in such a manner also that the ratios of the component velocities were constant in the process. If we write

$$F_1 dq_1 \ldots + F_n dq_n$$

for the moment of these forces, we have for the period of their action by equation (3)

$$\dot{p_1} = -\frac{d\epsilon_p}{dq_1} - \frac{d\epsilon_q}{dq_1} + F_1 = -\frac{d\epsilon}{dq_1} + F_1.$$

The work done by the force F_1 may be evaluated as follows:

$$\int F_1\, dq_1 = \int \dot{p_1}\, dq_1 + \int \frac{d\epsilon}{dq_1}\, dq_1,$$

where the last term may be cancelled because the configuration does not vary sensibly during the application of the forces. (It will be observed that the other terms contain factors which increase as the time of the action of the forces is diminished.) We have therefore,

$$\int F_1\, dq_1 = \int \dot{p_1}\, \dot{q_1}\, dt = \int \dot{q_1}\, dp = \frac{\dot{q_1}}{p_1} \int p_1\, dp_1. \qquad (131)$$

For since the p's are linear functions of the $\dot{q}$'s (with coefficients involving the q's) the supposed constancy of the q's and of the ratios of the $\dot{q}$'s will make the ratio $\dot{q_1}/p_1$ constant. The last integral is evidently to be taken between the limits zero and the value of p_1 in the phase originally considered, and the quantities before the integral sign may be taken as relating to that phase. We have therefore

$$\int F_1\, dq_1 = \tfrac{1}{2} p_1 \dot{q_1} = \tfrac{1}{2} p_1 \frac{d\epsilon}{dp_1}. \qquad (132)$$

That is: the several parts into which the kinetic energy is divided in equation (128) represent the amounts of energy communicated to the system by the several forces $F_1, \ldots F_n$ under the conditions mentioned.

The following transformation will not only give the value of the average kinetic energy, but will also serve to separate the distribution of the ensemble in configuration from its distribution in velocity.

Since $2\epsilon_p$ is a homogeneous quadratic function of the p's, which is incapable of a negative value, it can always be expressed (and in more than one way) as a sum of squares of

linear functions of the p's.* The coefficients in these linear functions, like those in the quadratic function, must be regarded in the general case as functions of the q's. Let

$$2\epsilon_p = u_1^2 + u_2^2 \ldots + u_n^2 \qquad (133)$$

where $u_1 \ldots u_n$ are such linear functions of the p's. If we write

$$\frac{d(p_1 \ldots p_n)}{d(u_1 \ldots u_n)}$$

for the Jacobian or determinant of the differential coefficients of the form dp/du, we may substitute

$$\frac{d(p_1 \ldots p_n)}{d(u_1 \ldots u_n)} du_1 \ldots du_n$$

for

$$dp_1 \ldots dp_n$$

under the multiple integral sign in any of our formulæ. It will be observed that this determinant is function of the q's alone. The sign of such a determinant depends on the relative order of the variables in the numerator and denominator. But since the suffixes of the u's are only used to distinguish these functions from one another, and no especial relation is supposed between a p and a u which have the same suffix, we may evidently, without loss of generality, suppose the suffixes so applied that the determinant is positive.

Since the u's are linear functions of the p's, when the integrations are to cover all values of the p's (for constant q's) once and only once, they must cover all values of the u's once and only once, and the limits will be $\pm\infty$ for all the u's. Without the supposition of the last paragraph the upper limits would not always be $+\infty$, as is evident on considering the effect of changing the sign of a u. But with the supposition which we have made (that the determinant is always positive) we may make the upper limits $+\infty$ and the lower $-\infty$ for all the u's. Analogous considerations will apply where the integrations do not cover all values of the p's and therefore of

* The reduction requires only the repeated application of the process of 'completing the square' used in the solution of quadratic equations.

the u's. The integrals may always be taken from a less to a greater value of a u.

The general integral which expresses the fractional part of the ensemble which falls within any given limits of phase is thus reduced to the form

$$\int \cdots \int e^{\frac{\psi-\epsilon_q}{\Theta}} \frac{d(p_1 \ldots p_n)}{d(u_1 \ldots u_n)} e^{-\frac{u_1^2 \ldots u_n^2}{2\Theta}} du_1 \ldots du_n dq_1 \ldots dq_n. \quad (134)$$

For the average value of the part of the kinetic energy which is represented by $\frac{1}{2}u_1^2$, whether the average is taken for the whole ensemble, or for a given configuration, we have therefore

$$\frac{1}{2}\bar{u}_1^2 = \frac{\int_{-\infty}^{+\infty} \frac{1}{2}u_1^2 e^{-\frac{u_1^2}{2\Theta}} du_1}{\int_{-\infty}^{+\infty} e^{-\frac{u_1^2}{2\Theta}} du_1} = \frac{(\frac{1}{2}\pi\Theta^3)^{\frac{1}{2}}}{(2\pi\Theta)^{\frac{1}{2}}} = \frac{\Theta}{2}, \quad (135)$$

and for the average of the whole kinetic energy, $\frac{1}{2}n\Theta$, as before.

The fractional part of the ensemble which lies within any given limits of *configuration*, is found by integrating (134) with respect to the u's from $-\infty$ to $+\infty$. This gives

$$(2\pi\Theta)^{\frac{n}{2}} \int \cdots \int e^{\frac{\psi-\epsilon_q}{\Theta}} \frac{d(p_1 \ldots p_n)}{d(u_1 \ldots u_n)} dq_1 \ldots dq_n \quad (136)$$

which shows that the value of the Jacobian is independent of the manner in which $2\epsilon_p$ is divided into a sum of squares. We may verify this directly, and at the same time obtain a more convenient expression for the Jacobian, as follows.

It will be observed that since the u's are linear functions of the p's, and the p's linear functions of the $\dot{q}$'s, the u's will be linear functions of the $\dot{q}$'s, so that a differential coefficient of the form du/dq will be independent of the $\dot{q}$'s, and function of the q's alone. Let us write dp_x/du_y for the general element of the Jacobian determinant. We have

$$\frac{dp_x}{du_y} = \frac{d}{du_y}\frac{d\epsilon}{d\dot{q}_x} = \frac{d}{du_y}\sum_{r=1}^{r=n}\frac{d\epsilon}{du_r}\frac{du_r}{d\dot{q}_x}$$

$$= \sum_{r=1}^{r=n}\left(\frac{d^2\epsilon}{du_y\,du_r}\frac{du_r}{d\dot{q}_x}\right) = \frac{d}{d\dot{q}_x}\frac{d\epsilon}{du_y} = \frac{du_y}{d\dot{q}_x} \qquad (137)$$

Therefore

$$\frac{d(p, \ldots p_n)}{d(u, \ldots u_n)} = \frac{d(u, \ldots u_n)}{d(\dot{q}, \ldots \dot{q}_n)} \qquad (138)$$

and

$$\left(\frac{d(p, \ldots p_n)}{d(u, \ldots u_n)}\right)^2 = \left(\frac{d(u, \ldots u_n)}{d(\dot{q}, \ldots \dot{q}_n)}\right)^2 = \frac{d(p, \ldots p_n)}{d(\dot{q}, \ldots \dot{q}_n)} \qquad (139)$$

These determinants are all functions of the q's alone.* The last is evidently the Hessian or determinant formed of the second differential coefficients of the kinetic energy with respect to $\dot{q}_1, \ldots \dot{q}_n$. We shall denote it by $\Delta_{\dot{q}}$. The reciprocal determinant

$$\frac{d(\dot{q}_1 \ldots \dot{q}_n)}{d(p_1 \ldots p_n)},$$

which is the Hessian of the kinetic energy regarded as function of the p's, we shall denote by Δ_p.

If we set

$$e^{-\frac{\psi_p}{\Theta}} = \int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{-\frac{\epsilon_p}{\Theta}}\,\Delta_p^{\frac{1}{2}}\,dp, \ldots dp_n$$

$$= \int_{-\infty}^{+\infty}\ldots\int_{-\infty}^{+\infty} e^{\frac{-u_1^2 \ldots -u_n^2}{2\Theta}}\,du_1 \ldots du_n = (2\pi\Theta)^{\frac{n}{2}}, \qquad (140)$$

and

$$\psi_q = \psi - \psi_p, \qquad (141)$$

* It will be observed that the proof of (137) depends on the *linear* relation between the u's and $\dot{q}$'s, which makes $\frac{du_r}{d\dot{q}_x}$ constant with respect to the differentiations here considered. Compare note on p. 12.

the fractional part of the ensemble which lies within any given limits of configuration (136) may be written

$$\int \cdots \int e^{\frac{\psi_q - \epsilon_q}{\Theta}} \Delta_q^{\frac{1}{2}} \, dq_1 \ldots dq_n, \tag{142}$$

where the constant ψ_q may be determined by the condition that the integral extended over all configurations has the value unity.*

* In the simple but important case in which Δ_q is independent of the q's, and ϵ_q a quadratic function of the q's, if we write ϵ_a for the least value of ϵ_q (or of ϵ) consistent with the given values of the external coördinates, the equation determining ψ_q may be written

$$e^{\frac{\epsilon_a - \psi_q}{\Theta}} = \Delta_q^{\frac{1}{2}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{(\epsilon_q - \epsilon_a)}{\Theta}} \, dq_1 \ldots dq_n.$$

If we denote by $q_1', \ldots q_n'$ the values of $q_1, \ldots q_n$ which give ϵ_q its least value ϵ_a, it is evident that $\epsilon_q - \epsilon_a$ is a homogenous quadratic function of the differences $q_1 - q_1'$, etc., and that $dq_1, \ldots dq_n$ may be regarded as the differentials of these differences. The evaluation of this integral is therefore analytically similar to that of the integral

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{\epsilon_p}{\Theta}} \, dp_1 \ldots dp_n,$$

for which we have found the value $\Delta_p^{-\frac{1}{2}} (2\pi\Theta)^{\frac{n}{2}}$. By the same method, or by analogy, we get

$$e^{\frac{\epsilon_a - \psi_q}{\Theta}} = \left(\frac{\Delta_q}{\Delta_q}\right)^{\frac{1}{2}} (2\pi\Theta)^{\frac{n}{2}},$$

where Δ_q is the Hessian of the potential energy as function of the q's. It will be observed that Δ_q depends on the forces of the system and is independent of the masses, while Δ_q or its reciprocal Δ_p depends on the masses and is independent of the forces. While each Hessian depends on the system of coördinates employed, the ratio Δ_q/Δ_q is the same for all systems.

Multiplying the last equation by (140), we have

$$e^{\frac{\epsilon_a - \psi}{\Theta}} = \left(\frac{\Delta_q}{\Delta_q}\right)^{\frac{1}{2}} (2\pi\Theta)^{n}.$$

For the average value of the potential energy, we have

$$\bar{\epsilon}_q - \epsilon_a = \frac{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} (\epsilon_q - \epsilon_a) \, e^{-\frac{\epsilon_q - \epsilon_a}{\Theta}} \, dq_1 \ldots dq_n}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{\epsilon_q - \epsilon_a}{\Theta}} \, dq_1 \ldots dq_n}.$$

When an ensemble of systems is distributed in *configuration* in the manner indicated in this formula, *i. e.*, when its distribution in configuration is the same as that of an ensemble canonically distributed in phase, we shall say, without any reference to its velocities, that it is *canonically distributed in configuration.*

For any given configuration, the fractional part of the systems which lie within any given limits of velocity is represented by the quotient of the multiple integral

$$\int \cdots \int e^{-\frac{\epsilon_p}{\Theta}} dp_1 \ldots dp_n,$$

or its equivalent

$$\int \cdots \int e^{\frac{-u_1^2 \ldots - u_n^2}{2\Theta}} \Delta_q^{\frac{1}{2}} du_1 \ldots du_n,$$

taken within those limits divided by the value of the same integral for the limits $\pm \infty$. But the value of the second multiple integral for the limits $\pm \infty$ is evidently

$$\Delta_q^{\frac{1}{2}} (2\pi\Theta)^{\frac{n}{2}}.$$

We may therefore write

$$\int \cdots \int e^{\frac{\psi_p - \epsilon_p}{\Theta}} du_1 \ldots du_n, \qquad (143)$$

The evaluation of this expression is similar to that of

$$\frac{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \epsilon_p e^{-\frac{\epsilon_p}{\Theta}} dp_1 \ldots dp_n}{\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{\epsilon_p}{\Theta}} dp_1 \ldots dp_n}$$

which expresses the average value of the kinetic energy, and for which we have found the value $\frac{1}{2} n \Theta$. We have accordingly

$$\bar{\epsilon}_q - \epsilon_a = \frac{1}{2} n \Theta.$$

Adding the equation

$$\bar{\epsilon}_p = \frac{1}{2} n \Theta,$$

we have

$$\bar{\epsilon} - \epsilon_a = n \Theta.$$

or
$$\int \cdots \int e^{\frac{\psi_p - \epsilon_p}{\Theta}} \Delta_p^{\frac{1}{2}} dp_1 \ldots dp_n, \tag{144}$$

or again
$$\int \cdots \int e^{\frac{\psi_p - \epsilon_p}{\Theta}} \Delta_{\dot{q}}^{\frac{1}{2}} d\dot{q}_1 \ldots d\dot{q}_n, \tag{145}$$

for the fractional part of the systems of any given configuration which lie within given limits of velocity.

When systems are distributed in velocity according to these formulae, *i. e.*, when the distribution in velocity is like that in an ensemble which is canonically distributed in phase, we shall say that they are *canonically distributed in velocity.*

The fractional part of the whole ensemble which falls within any given limits of phase, which we have before expressed in the form

$$\int \cdots \int e^{\frac{\psi - \epsilon}{\Theta}} dp_1 \ldots dp_n dq_1 \ldots dq_n, \tag{146}$$

may also be expressed in the form

$$\int \cdots \int e^{\frac{\psi - \epsilon}{\Theta}} \Delta_{\dot{q}} d\dot{q}_1 \ldots d\dot{q}_n dq_1 \ldots dq_n. \tag{147}$$

CHAPTER VI.

EXTENSION IN CONFIGURATION AND EXTENSION IN VELOCITY.

THE formulae relating to canonical ensembles in the closing paragraphs of the last chapter suggest certain general notions and principles, which we shall consider in this chapter, and which are not at all limited in their application to the canonical law of distribution.*

We have seen in Chapter IV. that the nature of the distribution which we have called canonical is independent of the system of coördinates by which it is described, being determined entirely by the modulus. It follows that the value represented by the multiple integral (142), which is the fractional part of the ensemble which lies within certain limiting configurations, is independent of the system of coördinates, being determined entirely by the limiting configurations with the modulus. Now ψ, as we have already seen, represents a value which is independent of the system of coördinates by which it is defined. The same is evidently true of ψ_p by equation (140), and therefore, by (141), of ψ_q. Hence the exponential factor in the multiple integral (142) represents a value which is independent of the system of coördinates. It follows that the value of a multiple integral of the form

$$\int \cdots \int \Delta_q^{\frac{1}{2}} dq_1 \cdots dq_n \qquad (148)$$

* These notions and principles are in fact such as a more logical arrangement of the subject would place in connection with those of Chapter I., to which they are closely related. The strict requirements of logical order have been sacrificed to the natural development of the subject, and very elementary notions have been left until they have presented themselves in the study of the leading problems.

is independent of the system of coördinates which is employed for its evaluation, as will appear at once, if we suppose the multiple integral to be broken up into parts so small that the exponential factor may be regarded as constant in each.

In the same way the formulae (144) and (145) which express the probability that a system (in a canonical ensemble) of given configuration will fall within certain limits of velocity, show that multiple integrals of the form

$$\int \dots \int \Delta_p^{\frac{1}{2}} \, dp_1 \dots dp_n \tag{149}$$

or

$$\int \dots \int \Delta_{\dot{q}}^{\frac{1}{2}} \, d\dot{q}_1 \dots d\dot{q}_n \tag{150}$$

relating to velocities possible for a given configuration, when the limits are formed by given velocities, have values independent of the system of coördinates employed.

These relations may easily be verified directly. It has already been proved that

$$\frac{d(P_1, \dots P_n)}{d(p_1, \dots p_n)} = \frac{d(\dot{q}_1 \dots \dot{q}_n)}{d(\dot{Q}_1, \dots \dot{Q}_n)} = \frac{d(q_1, \dots q_n)}{d(Q_1, \dots Q_n)}$$

where $q_1, \dots q_n, p_1, \dots p_n$ and $Q_1, \dots Q_n, P_1, \dots P_n$ are two systems of coördinates and momenta.* It follows that

$$\int \dots \int \left(\frac{d(p_1, \dots p_n)}{d(\dot{q}_1, \dots \dot{q}_n)} \right)^{\frac{1}{2}} dq_1 \dots dq_n$$

$$= \int \dots \int \left(\frac{d(p_1, \dots p_n)}{d(\dot{q}_1, \dots \dot{q}_n)} \right)^{\frac{1}{2}} \frac{d(q_1, \dots q_n)}{d(Q_1, \dots Q_n)} \, dQ_1 \dots dQ_n$$

$$= \int \dots \int \left(\frac{d(p_1, \dots p_n)}{d(\dot{q}_1, \dots \dot{q}_n)} \right)^{\frac{1}{2}} \left(\frac{d(P_1, \dots P_n)}{d(p_1, \dots p_n)} \right)^{\frac{1}{2}} \left(\frac{d(\dot{q}_1, \dots \dot{q}_n)}{d(\dot{Q}_1, \dots \dot{Q}_n)} \right)^{\frac{1}{2}} dQ_1 \dots dQ_n$$

$$= \int \dots \int \left(\frac{d(P_1, \dots P_n)}{d(\dot{Q}_1, \dots \dot{Q}_n)} \right) dQ_1 \dots dQ_n,$$

* See equation (29).

and

$$\int\cdots\int\left(\frac{d(\dot{Q}_1,\ldots\dot{Q}_n)}{d(P_1,\ldots P_n)}\right)^{\frac{1}{2}}dP_1\ldots P_n$$

$$=\int\cdots\int\left(\frac{d(\dot{Q}_1,\ldots\dot{Q}_n)}{d(P_1,\ldots P_n)}\right)^{\frac{1}{2}}\frac{d(P_1,\ldots P_n)}{d(p_1,\ldots p_n)}dp_1\ldots dp_n$$

$$=\int\cdots\int\left(\frac{d(\dot{Q}_1,\ldots\dot{Q}_n)}{d(P_1,\ldots P_n)}\right)^{\frac{1}{2}}\left(\frac{d(P_1,\ldots P_n)}{d(p_1,\ldots p_n)}\right)^{\frac{1}{2}}\left(\frac{d(\dot{q}_1,\ldots\dot{q}_n)}{d(\dot{Q}_1,\ldots\dot{Q}_n)}\right)^{\frac{1}{2}}dp_1\ldots dp_n$$

$$=\int\cdots\int\left(\frac{d(\dot{q}_1,\ldots\dot{q}_n)}{d(p_1,\ldots p_n)}\right)^{\frac{1}{2}}dp_1\ldots dp_n.$$

The multiple integral

$$\int\cdots\int dp_1\ldots dp_n dq_1\ldots dq_n, \tag{151}$$

which may also be written

$$\int\cdots\int\Delta_{\dot{q}}d\dot{q}_1\ldots d\dot{q}_n dq_1\ldots dq_n, \tag{152}$$

and which, when taken within any given limits of phase, has been shown to have a value independent of the coördinates employed, expresses what we have called an *extension-in-phase*.* In like manner we may say that the multiple integral (148) expresses an *extension-in-configuration*, and that the multiple integrals (149) and (150) express an *extension-in-velocity*. We have called

$$dp_1\ldots dp_n dq_1\ldots dq_n, \tag{153}$$

which is equivalent to

$$\Delta_{\dot{q}}d\dot{q}_1\ldots d\dot{q}_n dq_1\ldots dq_n, \tag{154}$$

an element of extension-in-phase. We may call

$$\Delta_{\dot{q}}^{\frac{1}{2}}dq_1\ldots dq_n \tag{155}$$

an element of extension-in-configuration, and

$$\Delta_{p}^{\frac{1}{2}}dp_1\ldots dp_n, \tag{156}$$

* See Chapter I, p. 10.

or its equivalent

$$\Delta_q^{\frac{1}{2}} d\dot{q}_1 \dots d\dot{q}_n, \tag{157}$$

an element of extension-in-velocity.

An extension-in-phase may always be regarded as an integral of elementary extensions-in-configuration multiplied each by an extension-in-velocity. This is evident from the formulae (151) and (152) which express an extension-in-phase, if we imagine the integrations relative to velocity to be first carried out.

The product of the two expressions for an element of extension-in-velocity (149) and (150) is evidently of the same dimensions as the product

$$p_1 \dots p_n \dot{q}_1 \dots \dot{q}_n$$

that is, as the nth power of energy, since every product of the form $p_1 \dot{q}_1$ has the dimensions of energy. Therefore an extension-in-velocity has the dimensions of the square root of the nth power of energy. Again we see by (155) and (156) that the product of an extension-in-configuration and an extension-in-velocity have the dimensions of the nth power of energy multiplied by the nth power of time. Therefore an extension-in-configuration has the dimensions of the nth power of time multiplied by the square root of the nth power of energy.

To the notion of extension-in-configuration there attach themselves certain other notions analogous to those which have presented themselves in connection with the notion of extension-in-phase. The number of systems of any ensemble (whether distributed canonically or in any other manner) which are contained in an element of extension-in-configuration, divided by the numerical value of that element, may be called the *density-in-configuration*. That is, if a certain configuration is specified by the coördinates $q_1 \dots q_n$, and the number of systems of which the coördinates fall between the limits q_1 and $q_1 + dq_1, \dots q_n$ and $q_n + dq_n$ is expressed by

$$D_q \Delta_q^{\frac{1}{2}} dq_1 \dots dq_n, \tag{158}$$

D_q will be the density-in-configuration. And if we set

$$e^{\eta_q} = \frac{D_q}{N}, \tag{159}$$

where N denotes, as usual, the total number of systems in the ensemble, the probability that an unspecified system of the ensemble will fall within the given limits of configuration, is expressed by

$$e^{\eta_q} \Delta_q^{\frac{1}{2}} dq_1 \dots dq_n. \tag{160}$$

We may call e^{η_q} the *coefficient of probability of the configuration*, and η_q the *index of probability of the configuration.*

The fractional part of the whole number of systems which are within any given limits of configuration will be expressed by the multiple integral

$$\int \dots \int e^{\eta_q} \Delta_q^{\frac{1}{2}} dq_1 \dots dq_n. \tag{161}$$

The value of this integral (taken within any given configurations) is therefore independent of the system of coördinates which is used. Since the same has been proved of the same integral without the factor e^{η_q}, it follows that the values of η_q and D_q for a given configuration in a given ensemble are independent of the system of coördinates which is used.

The notion of extension-in-velocity relates to systems having the same configuration.* If an ensemble is distributed both in configuration and in velocity, we may confine our attention to those systems which are contained within certain infinitesimal limits of configuration, and compare the whole number of such systems with those which are also contained

* Except in some simple cases, such as a system of material points, we cannot compare velocities in one configuration with velocities in another, and speak of their identity or difference except in a sense entirely artificial. We may indeed say that we call the velocities in one configuration the same as those in another when the quantities $\dot{q}_1, \dots \dot{q}_n$ have the same values in the two cases. But this signifies nothing until the system of coördinates has been defined. We might identify the velocities in the two cases which make the quantities $p_1, \dots p_n$ the same in each. This again would signify nothing independently of the system of coördinates employed.

within certain infinitesimal limits of velocity. The second of these numbers divided by the first expresses the probability that a system which is only specified as falling within the infinitesimal limits of configuration shall also fall within the infinitesimal limits of velocity. If the limits with respect to velocity are expressed by the condition that the momenta shall fall between the limits p_1 and $p_1 + dp_1, \ldots p_n$ and $p_n + dp_n$, the extension-in-velocity within those limits will be

$$\Delta_p^{\frac{1}{2}} dp_1 \ldots dp_n,$$

and we may express the probability in question by

$$e^{\eta_p} \Delta_p^{\frac{1}{2}} dp_1 \ldots dp_n. \tag{162}$$

This may be regarded as defining η_p.

The probability that a system which is only specified as having a configuration within certain infinitesimal limits shall also fall within any given limits of velocity will be expressed by the multiple integral

$$\int \ldots \int e^{\eta_p} \Delta_p^{\frac{1}{2}} dp_1 \ldots dp_n, \tag{163}$$

or its equivalent

$$\int \ldots \int e^{\eta_p} \Delta_{\dot{q}}^{\frac{1}{2}} d\dot{q}_1 \ldots d\dot{q}_n, \tag{164}$$

taken within the given limits.

It follows that the probability that the system will fall within the limits of velocity, $\dot{q}_1$ and $\dot{q}_1 + d\dot{q}_1, \ldots \dot{q}_n$ and $\dot{q}_n + d\dot{q}_n$ is expressed by

$$e^{\eta_p} \Delta_{\dot{q}}^{\frac{1}{2}} d\dot{q}_1 \ldots d\dot{q}_n. \tag{165}$$

The value of the integrals (163), (164) is independent of the system of coördinates and momenta which is used, as is also the value of the same integrals without the factor e^{η_p}; therefore the value of η_p must be independent of the system of coördinates and momenta. We may call e^{η_p} the *coefficient of probability of velocity*, and η_p the *index of probability of velocity*.

Comparing (160) and (162) with (40), we get

$$e^{\eta_q} e^{\eta_p} = P = e^{\eta} \tag{166}$$

or

$$\eta_q + \eta_p = \eta. \tag{167}$$

That is: the product of the coefficients of probability of configuration and of velocity is equal to the coefficient of probability of phase; the sum of the indices of probability of configuration and of velocity is equal to the index of probability of phase.

It is evident that e^{η_q} and e^{η_p} have the dimensions of the reciprocals of extension-in-configuration and extension-in-velocity respectively, *i. e.*, the dimensions of $t^{-n}\ \epsilon^{-\frac{n}{2}}$ and $\epsilon^{-\frac{n}{2}}$, where t represent any time, and ϵ any energy. If, therefore, the unit of time is multiplied by c_t, and the unit of energy by c_ϵ, every η_q will be increased by the addition of

$$n \log c_t + \tfrac{1}{2} n \log c_\epsilon, \tag{168}$$

and every η_p by the addition of

$$\tfrac{1}{2} n \log c_\epsilon.^* \tag{169}$$

It should be observed that the quantities which have been called *extension-in-configuration* and *extension-in-velocity* are not, as the terms might seem to imply, purely geometrical or kinematical conceptions. To express their nature more fully, they might appropriately have been called, respectively, *the dynamical measure of the extension in configuration*, and the *dynamical measure of the extension in velocity*. They depend upon the masses, although not upon the forces of the system. In the simple case of material points, where each point is limited to a given space, the extension-in-configuration is the product of the volumes within which the several points are confined (these may be the same or different), multiplied by the square root of the cube of the product of the masses of the several points. The extension-in-velocity for such systems is most easily defined as the extension-in-configuration of systems which have moved from the same configuration for the unit of time with the given velocities.

* Compare (47) in Chapter I.

In the general case, the notions of extension-in-configuration and extension-in-velocity may be connected as follows.

If an ensemble of similar systems of n degrees of freedom have the same configuration at a given instant, but are distributed throughout any finite extension-in-velocity, the same ensemble after an infinitesimal interval of time δt will be distributed throughout an extension in configuration equal to its original extension-in-velocity multiplied by δt^n.

In demonstrating this theorem, we shall write $q_1', \ldots q_n'$ for the initial values of the coördinates. The final values will evidently be connected with the initial by the equations

$$q_1 - q_1' = \dot{q}_1 \delta t, \ldots q_n - q_n' = \dot{q}_n \delta t. \tag{170}$$

Now the original extension-in-velocity is by definition represented by the integral

$$\int \ldots \int \Delta_{\dot{q}}^{\frac{1}{2}} d\dot{q}_1 \ldots d\dot{q}_n, \tag{171}$$

where the limits may be expressed by an equation of the form

$$F(\dot{q}_1, \ldots \dot{q}_n) = 0. \tag{172}$$

The same integral multiplied by the constant δt^n may be written

$$\int \ldots \int \Delta_{\dot{q}}^{\frac{1}{2}} d(\dot{q}_1 \delta t), \ldots d(\dot{q}_n \delta t), \tag{173}$$

and the limits may be written

$$F(\dot{q}_1 \ldots \dot{q}_n) = f(\dot{q}_1 \delta t, \ldots \dot{q}_n \delta t) = 0. \tag{174}$$

(It will be observed that δt as well as $\Delta_{\dot{q}}$ is constant in the integrations.) Now this integral is identically equal to

$$\int \ldots \int \Delta_{q}^{\frac{1}{2}} d(q_1 - q_1') \ldots d(q_n \ldots q_n'), \tag{175}$$

or its equivalent

$$\int \ldots \int \Delta_{q}^{\frac{1}{2}} dq_1 \ldots dq_n, \tag{176}$$

with limits expressed by the equation

$$f(q_1 - q_1', \ldots q_n - q_n') = 0. \tag{177}$$

But the systems which initially had velocities satisfying the equation (172) will after the interval δt have configurations satisfying equation (177). Therefore the extension-in-configuration represented by the last integral is that which belongs to the systems which originally had the extension-in-velocity represented by the integral (171).

Since the quantities which we have called extensions-in-phase, extensions-in-configuration, and extensions-in-velocity are independent of the nature of the system of coördinates used in their definitions, it is natural to seek definitions which shall be independent of the use of any coördinates. It will be sufficient to give the following definitions without formal proof of their equivalence with those given above, since they are less convenient for use than those founded on systems of coördinates, and since we shall in fact have no occasion to use them.

We commence with the definition of extension-in-velocity. We may imagine n independent velocities, $V_1, \ldots V_n$ of which a system in a given configuration is capable. We may conceive of the system as having a certain velocity V_0 combined with a part of each of these velocities $V_1 \ldots V_n$. By a part of V_1 is meant a velocity of the same nature as V_1 but in amount being anything between zero and V_1. Now all the velocities which may be thus described may be regarded as forming or lying in a certain extension of which we desire a measure. The case is greatly simplified if we suppose that certain relations exist between the velocities $V_1, \ldots V_n$, viz: that the kinetic energy due to any two of these velocities combined is the sum of the kinetic energies due to the velocities separately. In this case the extension-in-motion is the square root of the product of the doubled kinetic energies due to the n velocities $V_1, \ldots V_n$ taken separately.

The more general case may be reduced to this simpler case as follows. The velocity V_2 may always be regarded as composed of two velocities V_2' and V_2'', of which V_2' is of the same nature as V_1, (it may be more or less in amount, or opposite in sign,) while V_2'' satisfies the relation that the

kinetic energy due to V_1 and V_2'' combined is the sum of the kinetic energies due to these velocities taken separately. And the velocity V_3 may be regarded as compounded of three, V_3', V_3'', V_3''', of which V_3' is of the same nature as V_1, V_3'' of the same nature as V_2'', while V_3''' satisfies the relations that if combined either with V_1 or V_2'' the kinetic energy of the combined velocities is the sum of the kinetic energies of the velocities taken separately. When all the velocities $V_2, \ldots V_n$ have been thus decomposed, the square root of the product of the doubled kinetic energies of the several velocities V_1, V_2'', V_3''', etc., will be the value of the extension-in-velocity which is sought.

This method of evaluation of the extension-in-velocity which we are considering is perhaps the most simple and natural, but the result may be expressed in a more symmetrical form. Let us write ϵ_{12} for the kinetic energy of the velocities V_1 and V_2 combined, diminished by the sum of the kinetic energies due to the same velocities taken separately. This may be called the mutual energy of the velocities V_1 and V_2. Let the mutual energy of every pair of the velocities $V_1, \ldots V_n$ be expressed in the same way. Analogy would make ϵ_{11} represent the energy of twice V_1 diminished by twice the energy of V_1, *i. e.*, ϵ_{11} would represent twice the energy of V_1, although the term mutual energy is hardly appropriate to this case. At all events, let ϵ_{11} have this signification, and ϵ_{22} represent twice the energy of V_2, etc. The square root of the determinant

$$\begin{vmatrix} \epsilon_{11} & \epsilon_{12} & \ldots & \epsilon_{1n} \\ \epsilon_{21} & \epsilon_{22} & \ldots & \epsilon_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \epsilon_{n1} & \epsilon_{n2} & \ldots & \epsilon_{nn} \end{vmatrix}$$

represents the value of the extension-in-velocity determined as above described by the velocities $V_1, \ldots V_n$.

The statements of the preceding paragraph may be readily proved from the expression (157) on page 60, viz.,

$$\Delta_q^{\frac{1}{2}} \, d\dot{q}_1 \ldots d\dot{q}_n$$

by which the notion of an element of extension-in-velocity was

originally defined. Since $\Delta_{\dot{q}}$ in this expression represents the determinant of which the general element is

$$\frac{d^2\epsilon}{d\dot{q}_i d\dot{q}_j}$$

the square of the preceding expression represents the determinant of which the general element is

$$\frac{d^2\epsilon}{d\dot{q}_i d\dot{q}_j} d\dot{q}_i d\dot{q}_j.$$

Now we may regard the differentials of velocity $d\dot{q}_i$, $d\dot{q}_j$ as themselves infinitesimal velocities. Then the last expression represents the mutual energy of these velocities, and

$$\frac{d^2\epsilon}{d\dot{q}_i^2} d\dot{q}_i^2$$

represents twice the energy due to the velocity $d\dot{q}_i$.

The case which we have considered is an extension-in-velocity of the simplest form. All extensions-in-velocity do not have this form, but all may be regarded as composed of elementary extensions of this form, in the same manner as all volumes may be regarded as composed of elementary parallelepipeds.

Having thus a measure of extension-in-velocity founded, it will be observed, on the dynamical notion of kinetic energy, and not involving an explicit mention of coördinates, we may derive from it a measure of extension-in-configuration by the principle connecting these quantities which has been given in a preceding paragraph of this chapter.

The measure of extension-in-phase may be obtained from that of extension-in-configuration and of extension-in-velocity. For to every configuration in an extension-in-phase there will belong a certain extension-in-velocity, and the integral of the elements of extension-in-configuration within any extension-in-phase multiplied each by its extension-in-velocity is the measure of the extension-in-phase.

CHAPTER VII.

FARTHER DISCUSSION OF AVERAGES IN A CANONICAL ENSEMBLE OF SYSTEMS.

RETURNING to the case of a canonical distribution, we have for the index of probability of configuration

$$\eta_q = \frac{\psi_q - \epsilon_q}{\Theta} \tag{178}$$

as appears on comparison of formulae (142) and (161). It follows immediately from (142) that the average value in the ensemble of any quantity u which depends on the configuration alone is given by the formula

$$\bar{u} = \int_{\text{config.}}^{\text{all}} \cdots \int u\, e^{\frac{\psi_q - \epsilon_q}{\Theta}} \Delta_q^{\frac{1}{2}} dq_1 \ldots dq_n, \tag{179}$$

where the integrations cover all possible configurations. The value of ψ_q is evidently determined by the equation

$$e^{-\frac{\psi_q}{\Theta}} = \int_{\text{config.}}^{\text{all}} \cdots \int e^{-\frac{\epsilon_q}{\Theta}} \Delta_q^{\frac{1}{2}} dq_1 \ldots dq_n. \tag{180}$$

By differentiating the last equation we may obtain results analogous to those obtained in Chapter IV from the equation

$$e^{-\frac{\psi}{\Theta}} = \int_{\text{phases}}^{\text{all}} \cdots \int e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n.$$

As the process is identical, it is sufficient to give the results:

$$d\psi_q = \bar{\eta}_q d\Theta - \bar{A}_1 da_1 - \bar{A}_2 da_2 - \text{etc.}, \tag{181}$$

or, since
$$\psi_q = \bar{\epsilon}_q + \Theta \bar{\eta}_q, \tag{182}$$

and
$$d\psi_q = d\bar{\epsilon}_q + \bar{\eta}_q d\Theta + \Theta d\bar{\eta}_q, \tag{183}$$

$$d\bar{\epsilon}_q = -\Theta d\bar{\eta}_q - \bar{A}_1 da_1 - \bar{A}_2 da_2 - \text{etc.} \tag{184}$$

It appears from this equation that the differential relations subsisting between the average potential energy in an ensemble of systems canonically distributed, the modulus of distribution, the average index of probability of configuration, taken negatively, and the average forces exerted on external bodies, are equivalent to those enunciated by Clausius for the potential energy of a body, its temperature, a quantity which he called the disgregation, and the forces exerted on external bodies.*

For the index of probability of velocity, in the case of canonical distribution, we have by comparison of (144) and (163), or of (145) and (164),

$$\eta_p = \frac{\psi_p - \epsilon_p}{\Theta} \tag{185}$$

which gives
$$\bar{\eta}_p = \frac{\psi_p - \bar{\epsilon}_p}{\Theta}; \tag{186}$$

we have also
$$\bar{\epsilon}_p = \tfrac{1}{2} n \Theta, \tag{187}$$

and by (140),
$$\psi_p = -\tfrac{1}{2} n \Theta \log (2\pi\Theta). \tag{188}$$

From these equations we get by differentiation

$$d\psi_p = \bar{\eta}_p d\Theta, \tag{189}$$

and
$$d\bar{\epsilon}_p = -\Theta d\bar{\eta}_p. \tag{190}$$

The differential relation expressed in this equation between the average kinetic energy, the modulus, and the average index of probability of velocity, taken negatively, is identical with that given by Clausius *locis citatis* for the kinetic energy of a body, the temperature, and a quantity which he called the transformation-value of the kinetic energy.† The relations

$$\bar{\epsilon} = \bar{\epsilon}_q + \bar{\epsilon}_p, \qquad \bar{\eta} = \bar{\eta}_q + \bar{\eta}_p$$

* Pogg. Ann., Bd. CXVI, S. 73, (1862); ibid., Bd. CXXV, S. 353, (1865). See also Boltzmann, Sitzb. der Wiener Akad., Bd. LXIII, S. 728, (1871).

† Verwandlungswerth des Wärmeinhaltes.

are also identical with those given by Clausius for the corresponding quantities.

Equations (112) and (181) show that if ψ or ψ_q is known as function of Θ and a_1, a_2, etc., we can obtain by differentiation $\bar{\epsilon}$ or $\bar{\epsilon}_q$, and $\bar{A}_1$, $\bar{A}_2$, etc. as functions of the same variables. We have in fact

$$\bar{\epsilon} = \psi - \Theta \bar{\eta} = \psi - \Theta \frac{d\psi}{d\Theta}. \tag{191}$$

$$\bar{\epsilon}_q = \psi_q - \Theta \bar{\eta}_q = \psi_q - \Theta \frac{d\psi_q}{d\Theta}. \tag{192}$$

The corresponding equation relating to kinetic energy,

$$\bar{\epsilon}_p = \psi_p - \Theta \eta_p = \psi_p - \Theta \frac{d\psi_p}{d\Theta}. \tag{193}$$

which may be obtained in the same way, may be verified by the known relations (186), (187), and (188) between the variables. We have also

$$\bar{A}_1 = -\frac{d\psi}{da_1} = -\frac{d\psi_q}{da_1}, \tag{194}$$

etc., so that the average values of the external forces may be derived alike from ψ or from ψ_q.

The average values of the squares or higher powers of the energies (total, potential, or kinetic) may easily be obtained by repeated differentiations of ψ, ψ_q, ψ_p, or $\bar{\epsilon}$, $\bar{\epsilon}_q$, $\bar{\epsilon}_p$, with respect to Θ. By equation (108) we have

$$\bar{\epsilon} = \int_{\text{phases}}^{\text{all}} \cdots \int \epsilon e^{\frac{\psi - \epsilon}{\Theta}} dp_1 \ldots dq_n, \tag{195}$$

and differentiating with respect to Θ,

$$\frac{d\bar{\epsilon}}{d\Theta} = \int_{\text{phases}}^{\text{all}} \cdots \int \left(\frac{\epsilon^2 - \psi\epsilon}{\Theta^2} + \frac{\epsilon}{\Theta} \frac{d\psi}{d\Theta} \right) e^{\frac{\psi - \epsilon}{\Theta}} dp_1 \ldots dq_n, \tag{196}$$

whence, again by (108),

$$\frac{d\bar{\epsilon}}{d\Theta} = \frac{\bar{\epsilon}^2 - \bar{\psi}\bar{\epsilon}}{\Theta^2} + \frac{\bar{\epsilon}}{\Theta} \frac{d\psi}{d\Theta},$$

or
$$\overline{\epsilon^2} = \Theta^2 \frac{d\bar{\epsilon}}{d\Theta} + \bar{\epsilon}\left(\psi - \Theta \frac{d\psi}{d\Theta}\right). \tag{197}$$

Combining this with (191),

$$\overline{\epsilon^2} = \bar{\epsilon}^2 + \Theta^2 \frac{d\bar{\epsilon}}{d\Theta} = \left(\psi - \Theta \frac{d\psi}{d\Theta}\right)^2 - \Theta^3 \frac{d^2\psi}{d\Theta^2}. \tag{198}$$

In precisely the same way, from the equation

$$\bar{\epsilon}_q = \int_{\text{config.}}^{\text{all}} \cdots \int \epsilon_q e^{\frac{\psi_q - \epsilon_q}{\Theta}} \Delta_q^{\frac{1}{2}} dq_1 \ldots dq_n, \tag{199}$$

we may obtain

$$\overline{\epsilon_q^2} = \bar{\epsilon}_q^2 + \Theta^2 \frac{d\bar{\epsilon}_q}{d\Theta} = \left(\psi_q - \Theta \frac{d\psi_q}{d\Theta}\right)^2 - \Theta^3 \frac{d^2\psi_q}{d\Theta^2}. \tag{200}$$

In the same way also, if we confine ourselves to a particular configuration, from the equation

$$\bar{\epsilon}_p = \int_{\text{veloc.}}^{\text{all}} \cdots \int \epsilon_p e^{\frac{\psi_p - \epsilon_p}{\Theta}} \Delta_p^{\frac{1}{2}} dp_1 \ldots dp_n, \tag{201}$$

we obtain

$$\overline{\epsilon_p^2} = \bar{\epsilon}_p^2 + \Theta^2 \frac{d\epsilon_p}{d\Theta} = \left(\psi_p - \Theta \frac{d\psi_p}{d\Theta}\right)^2 - \Theta^3 \frac{d^2\psi_p}{d\Theta^2}, \tag{202}$$

which by (187) reduces to

$$\overline{\epsilon_p^2} = (\tfrac{1}{4} n^2 + \tfrac{1}{2} n)\, \Theta^2. \tag{203}$$

Since this value is independent of the configuration, we see that the average square of the kinetic energy for every configuration is the same, and therefore the same as for the whole ensemble. Hence $\overline{\epsilon_p^2}$ may be interpreted as the average either for any particular configuration, or for the whole ensemble. It will be observed that the value of this quantity is determined entirely by the modulus and the number of degrees of freedom of the system, and is in other respects independent of the nature of the system.

Of especial importance are the anomalies of the energies, or their deviations from their average values. The average value

of these anomalies is of course zero. The natural measure of such anomalies is the square root of their average square. Now

$$\overline{(\epsilon - \bar{\epsilon})^2} = \overline{\epsilon^2} - \bar{\epsilon}^2, \tag{204}$$

identically. Accordingly

$$\overline{(\epsilon - \bar{\epsilon})^2} = \Theta^2 \frac{d\bar{\epsilon}}{d\Theta} = -\Theta^3 \frac{d^2\psi}{d\Theta^2}. \tag{205}$$

In like manner,

$$\overline{(\epsilon_q - \bar{\epsilon}_q)^2} = \Theta^2 \frac{d\bar{\epsilon}_q}{d\Theta} = -\Theta^3 \frac{d^2\psi_q}{d\Theta^2}. \tag{206}$$

$$\overline{(\epsilon_p - \bar{\epsilon}_p)^2} = \Theta^2 \frac{d\bar{\epsilon}_p}{d\Theta} = -\Theta^3 \frac{d^2\psi_p}{d\Theta^2} = \tfrac{1}{2} n \Theta^2. \tag{207}$$

Hence

$$\overline{(\epsilon - \bar{\epsilon})^2} = \overline{(\epsilon_q - \bar{\epsilon}_q)^2} + \overline{(\epsilon_p - \bar{\epsilon}_p)^2}. \tag{208}$$

Equation (206) shows that the value of $d\bar{\epsilon}_q/d\Theta$ can never be negative, and that the value of $d^2\psi_q/d\Theta^2$ or $d\bar{\eta}_q/d\Theta$ can never be positive.*

To get an idea of the order of magnitude of these quantities, we may use the average kinetic energy as a term of comparison, this quantity being independent of the arbitrary constant involved in the definition of the potential energy. Since

* In the case discussed in the note on page 54, in which the potential energy is a quadratic function of the q's, and $\Delta_{\dot{q}}$ independent of the q's, we should get for the potential energy

$$\overline{(\epsilon_q - \bar{\epsilon}_q)^2} = \frac{1}{2} n \Theta^2,$$

and for the total energy

$$\overline{(\epsilon - \bar{\epsilon})^2} = n \Theta^2.$$

We may also write in this case,

$$\frac{\overline{(\epsilon_q - \bar{\epsilon}_q)^2}}{\overline{(\bar{\epsilon}_q - \epsilon_a)^2}} = \frac{2}{n},$$

$$\frac{\overline{(\epsilon - \bar{\epsilon})^2}}{\overline{(\bar{\epsilon} - \epsilon_a)^2}} = \frac{1}{n}.$$

$$\bar{\epsilon}_p = \tfrac{1}{2} n \Theta,$$

$$\frac{\overline{(\epsilon_p - \bar{\epsilon}_p)^2}}{\bar{\epsilon}_p^2} = \frac{2}{n}, \tag{209}$$

$$\frac{\overline{(\epsilon_q - \bar{\epsilon}_q)^2}}{\bar{\epsilon}_p^2} = \frac{2}{n} \frac{d\bar{\epsilon}_q}{d\bar{\epsilon}_p}, \tag{210}$$

$$\frac{\overline{(\epsilon - \bar{\epsilon})^2}}{\bar{\epsilon}_p^2} = \frac{2}{n} \frac{d\bar{\epsilon}}{d\bar{\epsilon}_p} = \frac{2}{n} + \frac{2}{n} \frac{d\bar{\epsilon}_q}{d\bar{\epsilon}_p}. \tag{211}$$

These equations show that when the number of degrees of freedom of the systems is very great, the mean squares of the anomalies of the energies (total, potential, and kinetic) are very small in comparison with the mean square of the kinetic energy, unless indeed the differential coefficient $d\bar{\epsilon}_q/d\bar{\epsilon}_p$ is of the same order of magnitude as n. Such values of $d\bar{\epsilon}_q/d\bar{\epsilon}_p$ can only occur within intervals $(\bar{\epsilon}_p'' - \bar{\epsilon}_p')$ which are of the order of magnitude of n^{-1}, unless it be in cases in which $\bar{\epsilon}_q$ is in general of an order of magnitude higher than $\bar{\epsilon}_p$. Postponing for the moment the consideration of such cases, it will be interesting to examine more closely the case of large values of $d\bar{\epsilon}_q/d\bar{\epsilon}_p$ within narrow limits. Let us suppose that for $\bar{\epsilon}_p'$ and $\bar{\epsilon}_p''$ the value of $d\bar{\epsilon}_q/d\bar{\epsilon}_p$ is of the order of magnitude of unity, but between these values of $\bar{\epsilon}_p$ very great values of the differential coefficient occur. Then in the ensemble having modulus Θ'' and average energies $\bar{\epsilon}_p''$ and $\bar{\epsilon}_q''$, values of ϵ_q sensibly greater than $\bar{\epsilon}_q''$ will be so rare that we may call them practically negligible. They will be still more rare in an ensemble of less modulus. For if we differentiate the equation

$$\eta_q = \frac{\psi_q - \epsilon_q}{\Theta}$$

regarding ϵ_q as constant, but Θ and therefore ψ_q as variable, we get

$$\left(\frac{d\eta_q}{d\Theta}\right)_{\epsilon_q} = \frac{1}{\Theta} \frac{d\psi_q}{d\Theta} - \frac{\psi_q - \epsilon_q}{\Theta^2}, \tag{212}$$

whence by (192)

$$\left(\frac{d\eta_q}{d\Theta}\right)_{\epsilon_q} = \frac{\epsilon_q - \bar{\epsilon}_q}{\Theta^2}. \tag{213}$$

That is, a diminution of the modulus will diminish the probability of all configurations for which the potential energy exceeds its average value in the ensemble. Again, in the ensemble having modulus Θ' and average energies $\bar{\epsilon}_p'$ and $\bar{\epsilon}_q'$, values of ϵ_q sensibly less than $\bar{\epsilon}_q'$ will be so rare as to be practically negligible. They will be still more rare in an ensemble of greater modulus, since by the same equation an increase of the modulus will diminish the probability of configurations for which the potential energy is less than its average value in the ensemble. Therefore, for values of Θ between Θ' and Θ'', and of $\bar{\epsilon}_p$ between $\bar{\epsilon}_p'$ and $\bar{\epsilon}_p''$, the individual values of ϵ_q will be practically limited to the interval between $\bar{\epsilon}_q'$ and $\bar{\epsilon}_q''$.

In the cases which remain to be considered, viz., when $d\bar{\epsilon}_q/d\bar{\epsilon}_p$ has very large values not confined to narrow limits, and consequently the differences of the mean potential energies in ensembles of different moduli are in general very large compared with the differences of the mean kinetic energies, it appears by (210) that the anomalies of mean square of potential energy, if not small in comparison with the mean kinetic energy, will yet in general be very small in comparison with differences of mean potential energy in ensembles having moderate differences of mean kinetic energy, — the exceptions being of the same character as described for the case when $d\bar{\epsilon}_q/d\bar{\epsilon}_p$ is not in general large.

It follows that to human experience and observation with respect to such an ensemble as we are considering, or with respect to systems which may be regarded as taken at random from such an ensemble, when the number of degrees of freedom is of such order of magnitude as the number of molecules in the bodies subject to our observation and experiment, $\epsilon - \bar{\epsilon}$, $\epsilon_p - \bar{\epsilon}_p$, $\epsilon_q - \bar{\epsilon}_q$ would be in general vanishing quantities, since such experience would not be wide enough to embrace the more considerable divergencies from the mean values, and such observation not nice enough to distinguish the ordinary divergencies. In other words, such ensembles would appear to human observation as ensembles of systems of uniform energy, and in which the potential and kinetic energies (sup-

posing that there were means of measuring these quantities separately) had each separately uniform values.* Exceptions might occur when for particular values of the modulus the differential coefficient $d\epsilon_q/d\bar{\epsilon}_p$ takes a very large value. To human observation the effect would be, that in ensembles in which Θ and $\bar{\epsilon}_p$ had certain critical values, $\bar{\epsilon}_q$ would be indeterminate within certain limits, viz., the values which would correspond to values of Θ and ϵ_p slightly less and slightly greater than the critical values. Such indeterminateness corresponds precisely to what we observe in experiments on the bodies which nature presents to us.†

To obtain general formulae for the average values of powers of the energies, we may proceed as follows. If h is any positive whole number, we have identically

$$\int_{\text{phases}}^{\text{all}} \cdots \int \epsilon^h e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n = \Theta^2 \frac{d}{d\Theta} \int_{\text{phases}}^{\text{all}} \cdots \int \epsilon^{h-1} e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n, \quad (214)$$

i. e., by (108),

$$\overline{\epsilon^h e^{-\frac{\psi}{\Theta}}} = \Theta^2 \frac{d}{d\Theta} \left(\overline{\epsilon^{h-1} e^{-\frac{\psi}{\Theta}}} \right). \quad (215)$$

Hence

$$\overline{\epsilon^h e^{-\frac{\psi}{\Theta}}} = \left(\Theta^2 \frac{d}{d\Theta} \right)^h e^{-\frac{\psi}{\Theta}}, \quad (216)$$

and

$$\overline{\epsilon^h} = e^{\frac{\psi}{\Theta}} \left(\Theta^2 \frac{d}{d\Theta} \right)^h e^{-\frac{\psi}{\Theta}}. \quad (217)$$

* This implies that the kinetic and potential energies of individual systems would each separately have values sensibly constant in time.

† As an example, we may take a system consisting of a fluid in a cylinder under a weighted piston, with a vacuum between the piston and the top of the cylinder, which is closed. The weighted piston is to be regarded as a part of the system. (This is formally necessary in order to satisfy the condition of the invariability of the external coördinates.) It is evident that at a certain temperature, viz., when the pressure of saturated vapor balances the weight of the piston, there is an indeterminateness in the values of the potential and total energies as functions of the temperature.

For $h = 1$, this gives

$$\bar{\epsilon} = -\Theta^2 \frac{d}{d\Theta}\left(\frac{\psi}{\Theta}\right) \tag{218}$$

which agrees with (191).
From (215) we have also

$$\overline{\epsilon^h} = \bar{\epsilon}\,\overline{\epsilon^{h-1}} + \Theta^2 \frac{d\overline{\epsilon^{h-1}}}{d\Theta} = \left(\bar{\epsilon} + \Theta^2 \frac{d}{d\Theta}\right)\epsilon^{h-1}, \tag{219}$$

$$\overline{\epsilon^h} = \left(\bar{\epsilon} + \Theta^2 \frac{d}{d\Theta}\right)^{h-1} \bar{\epsilon}. \tag{220}$$

In like manner from the identical equation

$$\int_{\text{config.}}^{\text{all}} \cdots \int \epsilon_q^h e^{-\frac{\epsilon_q}{\Theta}} \Delta_q^{\frac{1}{2}} dq_1 \ldots dq_n = \Theta^2 \frac{d}{d\Theta} \int_{\text{config.}}^{\text{all}} \cdots \int \epsilon_q^{h-1} e^{-\frac{\epsilon_q}{\Theta}} \Delta_q^{\frac{1}{2}} dq_1 \ldots dq_n, \tag{221}$$

we get

$$\overline{\epsilon_q^h} = e^{\frac{\psi_q}{\Theta}} \left(\Theta^2 \frac{d}{d\Theta}\right)^h e^{-\frac{\psi_q}{\Theta}}, \tag{222}$$

and

$$\overline{\epsilon_q^h} = \left(\bar{\epsilon}_q + \Theta^2 \frac{d}{d\Theta}\right)^{h-1} \bar{\epsilon}_q. \tag{223}$$

With respect to the kinetic energy similar equations will hold for averages taken for any particular configuration, or for the whole ensemble. But since

$$\bar{\epsilon}_p = \frac{n}{2}\Theta,$$

the equation

$$\overline{\epsilon_p^h} = \left(\bar{\epsilon}_p + \Theta^2 \frac{d}{d\Theta}\right)\overline{\epsilon_p^{h-1}} \tag{224}$$

reduces to

$$\overline{\epsilon_p^h} = \left(\frac{n}{2}\Theta + 2\Theta \frac{d}{d\Theta}\right)\overline{\epsilon^{h-1}} = \frac{n}{2}\left(\frac{n}{2}\Theta + \Theta^2 \frac{d}{d\Theta}\right)^{h-1}\Theta. \tag{225}$$

We have therefore

$$\overline{\epsilon_p^2} = \left(\frac{n}{2}+1\right)\frac{n}{2}\,\Theta^2. \tag{226}$$

$$\overline{\epsilon_p^3} = \left(\frac{n}{2}+2\right)\left(\frac{n}{2}+1\right)\frac{n}{2}\,\Theta^3. \tag{227}$$

$$\overline{\epsilon_p^h} = \frac{\Gamma\left(\frac{1}{2}n+h\right)}{\Gamma\left(\frac{1}{2}n\right)}\,\Theta^h. \tag{228}$$

The average values of the powers of the anomalies of the energies are perhaps most easily found as follows. We have identically, since ϵ is a function of Θ, while ϵ is a function of the p's and q's,

$$\Theta^2\frac{d}{d\Theta}\int_{\text{phases}}^{\text{all}}\cdots\int(\epsilon-\bar{\epsilon})^h e^{-\frac{\epsilon}{\Theta}}\,dp_1,\ldots dq_n =$$

$$\int_{\text{phases}}^{\text{all}}\cdots\int\left[\epsilon(\epsilon-\bar{\epsilon})^h - h(\epsilon-\bar{\epsilon})^{h-1}\Theta^2\frac{d\bar{\epsilon}}{d\Theta}\right]e^{-\frac{\epsilon}{\Theta}}\,dp_1,\ldots dq_n \tag{229}$$

i. e., by (108),

$$\Theta^2\frac{d}{d\Theta}\left[\overline{(\epsilon-\bar{\epsilon})^h}\,e^{-\frac{\psi}{\Theta}}\right] = \left[\overline{\epsilon(\epsilon-\bar{\epsilon})^h} - h\,\overline{(\epsilon-\bar{\epsilon})^{h-1}}\,\Theta^2\frac{d\bar{\epsilon}}{d\Theta}\right]e^{-\frac{\psi}{\Theta}}, \tag{230}$$

* In the case discussed in the note on page 54 we may easily get

$$\overline{(\epsilon_q-\epsilon_a)^h} = \left(\bar{\epsilon}_q-\epsilon_a+\Theta^2\frac{d}{d\Theta}\right)\overline{(\epsilon_q-\epsilon_a)^{h-1}},$$

which, with $\bar{\epsilon}_q-\epsilon_a=\frac{n}{2}\Theta$,

gives

$$\overline{(\epsilon_q-\epsilon_a)^h} = \left(\frac{n}{2}\Theta+\Theta^2\frac{d}{d\Theta}\right)\overline{(\epsilon_q-\epsilon_a)^{h-1}} = \frac{n}{2}\left(\frac{n}{2}\Theta+\Theta^2\frac{d}{d\Theta}\right)^{h-1}\Theta.$$

Hence $\overline{(\epsilon_q-\epsilon_a)^h}=\overline{\epsilon_p^h}$.

Again $\overline{(\epsilon-\epsilon_a)^h} = \left(\bar{\epsilon}-\epsilon_a+\Theta^2\frac{d}{d\Theta}\right)\overline{(\epsilon-\epsilon_a)^{h-1}}$,

which with $\bar{\epsilon}-\epsilon_a=n\Theta$

gives

$$\overline{(\epsilon-\epsilon_a)^h} = \left(n\Theta+\Theta^2\frac{d}{d\Theta}\right)\overline{(\epsilon-\epsilon_a)^{h-1}} = n\left(n\Theta+\Theta^2\frac{d}{d\Theta}\right)^{h-1}\Theta,$$

hence $\overline{(\epsilon-\epsilon_a)^h} = \frac{\Gamma(n+h)}{\Gamma(n)}\Theta^h$.

or since by (218)

$$\Theta^2 \frac{d}{d\Theta} e^{-\frac{\psi}{\Theta}} = \bar{\epsilon} e^{-\frac{\psi}{\Theta}},$$

$$\Theta^2 \frac{d}{d\Theta} \overline{(\epsilon - \bar{\epsilon})^h} + \overline{(\epsilon - \bar{\epsilon})^h \bar{\epsilon}} = \overline{\epsilon (\epsilon - \bar{\epsilon})^h} - h \overline{(\epsilon - \bar{\epsilon})^{h-1}} \Theta^2 \frac{d\bar{\epsilon}}{d\Theta},$$

$$\overline{(\epsilon - \bar{\epsilon})^{h+1}} = \Theta^2 \frac{d}{d\Theta} \overline{(\epsilon - \bar{\epsilon})^h} + h \overline{(\epsilon - \bar{\epsilon})^{h-1}} \Theta^2 \frac{d\bar{\epsilon}}{d\Theta}. \quad (231)$$

In precisely the same way we may obtain for the potential energy

$$\overline{(\epsilon_q - \bar{\epsilon}_q)^{h+1}} = \Theta^2 \frac{d}{d\Theta} \overline{(\epsilon_q - \bar{\epsilon}_q)^h} + h \overline{(\epsilon_q - \bar{\epsilon}_q)^{h-1}} \Theta^2 \frac{d\bar{\epsilon}_q}{d\Theta}. \quad (232)$$

By successive applications of (231) we obtain

$$\overline{(\epsilon - \bar{\epsilon})^2} = D\bar{\epsilon}$$

$$\overline{(\epsilon - \bar{\epsilon})^3} = D^2\bar{\epsilon}$$

$$\overline{(\epsilon - \bar{\epsilon})^4} = D^3\bar{\epsilon} + 3(D\bar{\epsilon})^2$$

$$\overline{(\epsilon - \bar{\epsilon})^5} = D^4\bar{\epsilon} + 10 D\bar{\epsilon} D^2\bar{\epsilon}$$

$$\overline{(\epsilon - \bar{\epsilon})^6} = D^5\bar{\epsilon} + 15 D\bar{\epsilon} D^3\bar{\epsilon} + 10(D^2\bar{\epsilon})^2 + 15(D\bar{\epsilon})^3 \text{ etc.}$$

where D represents the operator $\Theta^2 d/d\Theta$. Similar expressions relating to the potential energy may be derived from (232).

For the kinetic energy we may write similar equations in which the averages may be taken either for a single configuration or for the whole ensemble. But since

$$\frac{d\epsilon_p}{d\Theta} = \frac{n}{2}$$

the general formula reduces to

$$\overline{(\epsilon_p - \bar{\epsilon}_p)^{h+1}} = \Theta^2 \frac{d}{d\Theta} \overline{(\epsilon_p - \bar{\epsilon}_p)^h} + \tfrac{1}{2} n h \Theta^2 \overline{(\epsilon_p - \bar{\epsilon}_p)^{h-1}} \quad (233)$$

or

$$\frac{\overline{(\epsilon_p - \bar{\epsilon}_p)^{h+1}}}{\bar{\epsilon}_p^{h+1}} = \frac{2\Theta}{n} \frac{d}{d\Theta} \frac{\overline{(\epsilon_p - \bar{\epsilon}_p)^h}}{\bar{\epsilon}_p^h} + \frac{2h}{n} \frac{\overline{(\epsilon_p - \bar{\epsilon}_p)^h}}{\bar{\epsilon}_p^h} + \frac{2h}{n} \frac{\overline{(\epsilon_p - \bar{\epsilon}_p)^{h-1}}}{\bar{\epsilon}_p^{h-1}} \quad (234)$$

But since identically

$$\overline{\frac{(\epsilon_p - \bar{\epsilon}_p)^0}{\bar{\epsilon}_p^{\,0}}} = 1, \qquad \overline{\frac{(\epsilon - \bar{\epsilon})^1}{\bar{\epsilon}}} = 0,$$

the value of the corresponding expression for any index will be independent of Θ and the formula reduces to

$$\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{h+1}} = \frac{2h}{n}\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{h}} + \frac{2h}{n}\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{h-1}} \qquad (235)$$

we have therefore

$$\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{0}} = 1, \qquad \overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{3}} = \frac{8}{n^2},$$

$$\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{1}} = 0, \qquad \overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{4}} = \frac{48}{n^3} + \frac{12}{n^2},$$

$$\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^{2}} = \frac{2}{n}, \qquad \text{etc.*}$$

It will be observed that when ψ or $\bar{\epsilon}$ is given as function of Θ, all averages of the form $\overline{\epsilon^h}$ or $\overline{(\epsilon - \bar{\epsilon})^h}$ are thereby deter-

* In the case discussed in the preceding foot-notes we get easily

$$\overline{(\epsilon_q - \bar{\epsilon}_q)^h} = \overline{(\epsilon_p - \bar{\epsilon}_p)^h},$$

and

$$\overline{\left(\frac{\epsilon_q - \bar{\epsilon}_q}{\bar{\epsilon}_q - \epsilon_a}\right)^h} = \overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^h}.$$

For the total energy we have in this case

$$\overline{\left(\frac{\epsilon - \bar{\epsilon}}{\bar{\epsilon} - \epsilon_a}\right)^{h+1}} = \frac{h}{n}\overline{\left(\frac{\epsilon - \bar{\epsilon}}{\bar{\epsilon} - \epsilon_a}\right)^{h}} + \frac{h}{n}\overline{\left(\frac{\epsilon - \bar{\epsilon}}{\bar{\epsilon} - \epsilon_a}\right)^{h-1}}.$$

$$\overline{\left(\frac{\epsilon - \bar{\epsilon}}{\bar{\epsilon} - \epsilon_a}\right)^{2}} = \frac{1}{n}, \qquad \overline{\left(\frac{\epsilon - \bar{\epsilon}}{\bar{\epsilon} - \epsilon_a}\right)^{4}} = \frac{3}{n^2} + \frac{6}{n^3},$$

$$\overline{\left(\frac{\epsilon - \bar{\epsilon}}{\bar{\epsilon} - \epsilon_a}\right)^{3}} = \frac{2}{n^2}, \qquad \text{etc.}$$

mined. So also if ψ_q or $\bar{\epsilon}_q$ is given as function of Θ, all averages of the form $\overline{\bar{\epsilon}_q^h}$ or $\overline{(\epsilon_q - \bar{\epsilon}_q)^h}$ are determined. But

$$\bar{\epsilon}_q = \bar{\epsilon} - \tfrac{1}{2} n\Theta.$$

Therefore if any one of the quantities ψ, ψ_q, $\bar{\epsilon}$, $\bar{\epsilon}_q$ is known as function of Θ, and n is also known, all averages of any of the forms mentioned are thereby determined as functions of the same variable. In any case all averages of the form

$$\overline{\left(\frac{\epsilon_p - \bar{\epsilon}_p}{\bar{\epsilon}_p}\right)^h}$$

are known in terms of n alone, and have the same value whether taken for the whole ensemble or limited to any particular configuration.

If we differentiate the equation

$$\int_{\text{phases}}^{\text{all}} \cdots \int e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n = 1 \tag{236}$$

with respect to a_1, and multiply by Θ, we have

$$\int \cdots \int \left[\frac{d\psi}{da_1} - \frac{d\epsilon}{da_1}\right] e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n = 0. \tag{237}$$

Differentiating again, with respect to a_1, with respect to a_2, and with respect to Θ, we have

$$\int \cdots \int \left[\frac{d^2\psi}{da_1^2} - \frac{d^2\epsilon}{da_1^2} + \frac{1}{\Theta}\left(\frac{d\psi}{da_1} - \frac{d\epsilon}{da_1}\right)^2\right] e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n = 0, \tag{238}$$

$$\int \cdots \int \left[\frac{d^2\psi}{da_1 da_2} - \frac{d^2\epsilon}{da_1 da_2} + \frac{1}{\Theta}\left(\frac{d\psi}{da_1} - \frac{d\epsilon}{da_1}\right)\left(\frac{d\psi}{da_2} - \frac{d\epsilon}{da_2}\right)\right] e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n = 0, \tag{239}$$

$$\int \cdots \int \left[\frac{d^2\psi}{da_1 d\Theta} + \left(\frac{d\psi}{da_1} - \frac{d\epsilon}{da_1}\right)\left(\frac{1}{\Theta}\frac{d\psi}{d\Theta} - \frac{\psi-\epsilon}{\Theta^2}\right)\right] e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n = 0. \tag{240}$$

The multiple integrals in the last four equations represent the average values of the expressions in the brackets, which we may therefore set equal to zero. The first gives

$$\frac{d\psi}{da_1} = \frac{\overline{d\epsilon}}{da_1} = -\bar{A}_1, \tag{241}$$

as already obtained. With this relation and (191) we get from the other equations

$$\overline{(A_1 - \bar{A}_1)^2} = \Theta\left(\frac{\overline{d^2\epsilon}}{da_1^2} - \frac{d^2\psi}{da_1^2}\right) = \Theta\left(\frac{d\bar{A}_1}{da_1} - \frac{\overline{dA_1}}{da_1}\right) \tag{242}$$

$$\begin{aligned}\overline{(A_1 - \bar{A}_1)(A_2 - \bar{A}_2)} &= \Theta\left(\frac{\overline{d^2\epsilon}}{da_1\,da_2} - \frac{d^2\psi}{da_1\,da_2}\right) \\ &= \Theta\left(\frac{d\bar{A}_1}{da_2} - \frac{\overline{dA_1}}{da_2}\right) = \Theta\left(\frac{d\bar{A}_2}{da_1} - \frac{\overline{dA_2}}{da_1}\right)\end{aligned} \tag{243}$$

$$\overline{(A_1 - \bar{A}_1)(\epsilon - \bar{\epsilon})} = -\Theta^2\frac{d^2\psi}{da_1\,d\Theta} = \Theta^2\frac{d\bar{A}_1}{d\Theta} = -\Theta^2\frac{d\bar{\eta}}{da_1}.$$

We may add for comparison equation (205), which might be derived from (236) by differentiating twice with respect to Θ:

$$\overline{(\epsilon - \bar{\epsilon})^2} = -\Theta^3\frac{d^2\psi}{d\Theta^2} = \Theta^2\frac{d\bar{\epsilon}}{d\Theta}. \tag{244}$$

The two last equations give

$$\overline{(A_1 - \bar{A}_1)(\epsilon - \bar{\epsilon})} = \frac{d\bar{A}}{d\bar{\epsilon}}\overline{(\epsilon - \bar{\epsilon})^2}. \tag{245}$$

If ψ or $\bar{\epsilon}$ is known as function of Θ, a_1, a_2, etc., $\overline{(\epsilon - \bar{\epsilon})^2}$ may be obtained by differentiation as function of the same variables. And if ψ, or $\bar{A}_1$, or $\bar{\eta}$ is known as function of Θ, a_1, etc., $\overline{(A_1 - \bar{A}_1)(\epsilon - \bar{\epsilon})}$ may be obtained by differentiation. But $\overline{(A_1 - \bar{A}_1)^2}$ and $\overline{(A_1 - \bar{A}_1)(A_2 - \bar{A}_2)}$ cannot be obtained in any similar manner. We have seen that $\overline{(\epsilon - \bar{\epsilon})^2}$ is in general a vanishing quantity for very great values of n, which we may regard as contained implicitly in Θ as a divisor. The same is true of $\overline{(A_1 - \bar{A}_1)(\epsilon - \bar{\epsilon})}$. It does not appear that we can assert the same of $\overline{(A_1 - \bar{A}_1)^2}$ or $\overline{(A_1 - \bar{A}_1)(A_2 - \bar{A}_2)}$, since

$d^2\epsilon/da_1^2$ may be very great. The quantities $d^2\epsilon/da_1^2$ and $d^2\psi/da_1^2$ belong to the class called elasticities. The former expression represents an elasticity measured under the condition that while a_1 is varied the internal coördinates $q_1, \ldots q_n$ all remain fixed. The latter is an elasticity measured under the condition that when a_1 is varied the ensemble remains canonically distributed within the same modulus. This corresponds to an elasticity in physics measured under the condition of constant temperature. It is evident that the former is greater than the latter, and it may be enormously greater.

The divergences of the force A_1 from its average value are due in part to the differences of energy in the systems of the ensemble, and in part to the differences in the value of the forces which exist in systems of the same energy. If we write $\overline{A_1}\rceil_\epsilon$ for the average value of A_1 in systems of the ensemble which have any same energy, it will be determined by the equation

$$\overline{A_1}\rceil_\epsilon = \frac{\int \cdots \int -\frac{d\epsilon}{da_1} e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n}{\int \cdots \int e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n} \tag{246}$$

where the limits of integration in both multiple integrals are two values of the energy which differ infinitely little, say ϵ and $\epsilon + d\epsilon$. This will make the factor $e^{\frac{\psi-\epsilon}{\Theta}}$ constant within the limits of integration, and it may be cancelled in the numerator and denominator, leaving

$$\overline{A_1}\rceil_\epsilon = \frac{\int \cdots \int -\frac{d\epsilon}{da_1} dp_1 \ldots dq_n}{\int \cdots \int dp_1 \ldots dq_n} \tag{247}$$

where the integrals as before are to be taken between ϵ and $\epsilon + d\epsilon$. $\overline{A_1}\rceil_\epsilon$ is therefore independent of Θ, being a function of the energy and the external coördinates.

Now we have identically

$$A_1 - \bar{A}_1 = (A_1 - \overline{A_1}|_\epsilon) + (\overline{A_1}|_\epsilon - \bar{A}_1),$$

where $A_1 - \overline{A_1}|_\epsilon$ denotes the excess of the force (tending to increase a_1) exerted by any system above the average of such forces for systems of the same energy. Accordingly,

$$\overline{(A_1 - \bar{A}_1)^2} = \overline{(A_1 - \overline{A_1}|_\epsilon)^2} + 2\,\overline{(A_1 - \overline{A_1}|_\epsilon)(\overline{A_1}|_\epsilon - \bar{A}_1)} + \overline{(\overline{A_1}|_\epsilon - \bar{A}_1)^2}.$$

But the average value of $(A_1 - \overline{A_1}|_\epsilon)(\overline{A_1}|_\epsilon - \bar{A}_1)$ for systems of the ensemble which have the same energy is zero, since for such systems the second factor is constant. Therefore the average for the whole ensemble is zero, and

$$\overline{(A_1 - \bar{A}_1)^2} = \overline{(A_1 - \overline{A_1}|_\epsilon)^2} + \overline{(\overline{A_1}|_\epsilon - A_1)^2}. \tag{248}$$

In the same way it may be shown that

$$\overline{(A_1 - A_1)(\epsilon - \bar{\epsilon})} = \overline{(\overline{A_1}|_\epsilon - \bar{A}_1)(\epsilon - \bar{\epsilon})}. \tag{249}$$

It is evident that in ensembles in which the anomalies of energy $\epsilon - \bar{\epsilon}$ may be regarded as insensible the same will be true of the quantities represented by $\overline{A_1}|_\epsilon - \bar{A}_1$.

The properties of quantities of the form $\overline{A_1}|_\epsilon$ will be farther considered in Chapter X, which will be devoted to ensembles of constant energy.

It may not be without interest to consider some general formulae relating to averages in a canonical ensemble, which embrace many of the results which have been given in this chapter.

Let u be any function of the internal and external coördinates with the momenta and modulus. We have by definition

$$\bar{u} = \int_{\text{phases}}^{\text{all}} \cdots \int u e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \ldots dq_n \tag{250}$$

If we differentiate with respect to Θ, we have

$$\frac{d\bar{u}}{d\Theta} = \int_{\text{phases}}^{\text{all}} \cdots \int \left(\frac{du}{d\Theta} - \frac{u}{\Theta^2}(\psi - \epsilon) + \frac{u}{\Theta}\frac{d\psi}{d\Theta}\right) e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \ldots dq_n$$

or
$$\frac{d\bar{u}}{d\Theta} = \overline{\frac{du}{d\Theta}} - \overline{\frac{u(\psi - \epsilon)}{\Theta^2}} + \frac{\bar{u}}{\Theta}\frac{d\psi}{d\Theta}. \tag{251}$$

Setting $u = 1$ in this equation, we get

$$\frac{d\psi}{d\Theta} = \frac{\psi - \bar{\epsilon}}{\Theta},$$

and substituting this value, we have

$$\frac{d\bar{u}}{d\Theta} = \overline{\frac{du}{d\Theta}} + \overline{\frac{u\epsilon}{\Theta^2}} - \frac{\bar{u}\bar{\epsilon}}{\Theta^2}$$

or
$$\Theta^2\frac{d\bar{u}}{d\Theta} - \Theta^2\overline{\frac{du}{d\Theta}} = \overline{u\epsilon} - \bar{u}\bar{\epsilon} = \overline{(u - \bar{u})(\epsilon - \bar{\epsilon})}. \tag{252}$$

If we differentiate equation (250) with respect to a (which may represent any of the external coördinates), and write A for the force $-\frac{d\epsilon}{dA}$, we get

$$\frac{d\bar{u}}{da} = \int_{\text{phases}}^{\text{all}} \cdots \int \left(\frac{du}{da} + \frac{u}{\Theta}\frac{d\psi}{da} + \frac{u}{\Theta}A\right) e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \ldots dq_n$$

or
$$\frac{d\bar{u}}{da} = \overline{\frac{du}{da}} + \frac{\bar{u}}{\Theta}\frac{d\psi}{da} + \overline{\frac{uA}{\Theta}} \tag{253}$$

Setting $u = 1$ in this equation, we get

$$\frac{d\psi}{da} = -\bar{A}.$$

Substituting this value, we have

$$\frac{d\bar{u}}{da} = \overline{\frac{du}{da}} + \overline{\frac{uA}{\Theta}} - \frac{\bar{u}\bar{A}}{\Theta}, \tag{254}$$

or
$$\Theta\frac{d\bar{u}}{da} - \Theta\overline{\frac{du}{da}} = \overline{uA} - \bar{u}\bar{A} = \overline{(u - \bar{u})(A - \bar{A})}. \tag{255}$$

Repeated applications of the principles expressed by equations (252) and (255) are perhaps best made in the particular cases. Yet we may write (252) in this form

$$\overline{(\epsilon + D)\,(u - \bar{u})} = 0, \tag{256}$$

where D represents the operator $\Theta^2\, d/d\Theta$.
Hence

$$\overline{(\epsilon + D)^h\,(u - \bar{u})} = 0, \tag{257}$$

where h is any positive whole number. It will be observed, that since ϵ is not function of Θ, $(\epsilon + D)^h$ may be expanded by the binomial theorem. Or, we may write

$$\overline{(\epsilon + D)\,u} = (\bar{\epsilon} + D)\,\bar{u}, \tag{258}$$

whence

$$\overline{(\epsilon + D)^h\,u} = (\bar{\epsilon} + D)^h\,\bar{u}. \tag{259}$$

But the operator $(\bar{\epsilon} + D)^h$, although in some respects more simple than the operator without the average sign on the ϵ, cannot be expanded by the binomial theorem, since $\bar{\epsilon}$ is a function of Θ with the external coördinates.

So from equation (254) we have

$$\overline{\left(\frac{A}{\Theta} + \frac{d}{da}\right)(u - \bar{u})} = 0, \tag{260}$$

whence

$$\overline{\left(\frac{A}{\Theta} + \frac{d}{da}\right)^h (u - \bar{u})} = 0; \tag{261}$$

and

$$\overline{\left(\frac{A}{\Theta} + \frac{d}{da}\right) u} = \left(\frac{\bar{A}}{\Theta} + \frac{d}{da}\right)\bar{u}, \tag{262}$$

whence

$$\overline{\left(\frac{A}{\Theta} + \frac{d}{da}\right)^h u} = \left(\frac{\bar{A}}{\Theta} + \frac{d}{da}\right)^h \bar{u}. \tag{263}$$

The binomial theorem cannot be applied to these operators.

Again, if we now distinguish, as usual, the several external coördinates by suffixes, we may apply successively to the expression $u - \bar{u}$ any or all of the operators

$$\epsilon + \Theta^2 \frac{d}{d\Theta}, \quad A_1 + \Theta \frac{d}{da_1}, \quad A_2 + \Theta \frac{d}{da_2}, \quad \text{etc.} \tag{264}$$

as many times as we choose, and in any order, the average value of the result will be zero. Or, if we apply the same operators to u, and finally take the average value, it will be the same as the value obtained by writing the sign of average separately as u, and on ϵ, A_1, A_2, etc., in all the operators.

If u is independent of the momenta, formulae similar to the preceding, but having ϵ_q in place of ϵ, may be derived from equation (179).

CHAPTER VIII.

ON CERTAIN IMPORTANT FUNCTIONS OF THE ENERGIES OF A SYSTEM.

In order to consider more particularly the distribution of a canonical ensemble in energy, and for other purposes, it will be convenient to use the following definitions and notations.

Let us denote by V the extension-in-phase below a certain limit of energy which we shall call ϵ. That is, let

$$V = \int \dots \int dp_1 \dots dq_n, \tag{265}$$

the integration being extended (with constant values of the external coördinates) over all phases for which the energy is less than the limit ϵ. We shall suppose that the value of this integral is not infinite, except for an infinite value of the limiting energy. This will not exclude any kind of system to which the canonical distribution is applicable. For if

$$\int \dots \int e^{-\frac{\epsilon}{\Theta}} dp_1 \dots dq_n$$

taken without limits has a finite value,* the less value represented by

$$e^{-\frac{\epsilon}{\Theta}} \int \dots \int dp_1 \dots dq_n$$

taken below a limiting value of ϵ, and with the ϵ before the integral sign representing that limiting value, will also be finite. Therefore the value of V, which differs only by a constant factor, will also be finite, for finite ϵ. It is a function of ϵ and the external coördinates, a continuous increasing

* This is a necessary condition of the canonical distribution. See Chapter IV, p. 35.

function of ϵ, which becomes infinite with ϵ, and vanishes for the smallest possible value of ϵ, or for $\epsilon = -\infty$, if the energy may be diminished without limit.

Let us also set

$$\phi = \log \frac{dV}{d\epsilon}. \tag{266}$$

The extension in phase between any two limits of energy, ϵ' and ϵ'', will be represented by the integral

$$\int_{\epsilon'}^{\epsilon''} e^{\phi}\, d\epsilon. \tag{267}$$

And in general, we may substitute $\epsilon^{\phi}\, d\epsilon$ for $dp_1 \ldots dq_n$ in a $2n$-fold integral, reducing it to a simple integral, whenever the limits can be expressed by the energy alone, and the other factor under the integral sign is a function of the energy alone, or with quantities which are constant in the integration.

In particular we observe that the probability that the energy of an unspecified system of a canonical ensemble lies between the limits ϵ' and ϵ'' will be represented by the integral *

$$\int_{\epsilon'}^{\epsilon''} e^{\frac{\psi-\epsilon}{\Theta}+\phi}\, d\epsilon, \tag{268}$$

and that the average value in the ensemble of any quantity which only varies with the energy is given by the equation †

$$\bar{u} = \int_{V=0}^{\epsilon=\infty} u e^{\frac{\psi-\epsilon}{\Theta}+\phi}\, d\epsilon, \tag{269}$$

where we may regard the constant ψ as determined by the equation ‡

$$e^{-\frac{\psi}{\Theta}} = \int_{V=0}^{\epsilon=\infty} e^{-\frac{\epsilon}{\Theta}+\phi}\, d\epsilon, \tag{270}$$

In regard to the lower limit in these integrals, it will be observed that $V = 0$ is equivalent to the condition that the value of ϵ is the least possible.

* Compare equation (93). † Compare equation (108).

‡ Compare equation (92).

In like manner, let us denote by V_q the extension-in-configuration below a certain limit of potential energy which we may call ϵ_q. That is, let

$$V_q = \int \dots \int \Delta_q^{\frac{1}{2}} dq_1 \dots dq_n, \tag{271}$$

the integration being extended (with constant values of the external coördinates) over all configurations for which the potential energy is less than ϵ_q. V_q will be a function of ϵ_q with the external coördinates, an increasing function of ϵ_q, which does not become infinite (in such cases as we shall consider*) for any finite value of ϵ_q. It vanishes for the least possible value of ϵ_q, or for $\epsilon_q = -\infty$, if ϵ_q can be diminished without limit. It is not always a continuous function of ϵ_q. In fact, if there is a finite extension-in-configuration of constant potential energy, the corresponding value of V_q will not include that extension-in-configuration, but if ϵ_q be increased infinitesimally, the corresponding value of V_q will be increased by that finite extension-in-configuration.

Let us also set

$$\phi_q = \log \frac{d V_q}{d \epsilon_q}. \tag{272}$$

The extension-in-configuration between any two limits of potential energy ϵ_q' and ϵ_q'' may be represented by the integral

$$\int_{\epsilon_q'}^{\epsilon_q''} e^{\phi_q} d\epsilon_q \tag{273}$$

whenever there is no discontinuity in the value of V_q as function of ϵ_q between or at those limits, that is, whenever there is no finite extension-in-configuration of constant potential energy between or at the limits. And in general, with the restriction mentioned, we may substitute $e^{\phi_q} d\epsilon_q$ for $\Delta_q^{\frac{1}{2}} dq_1 \dots dq_n$ in an n-fold integral, reducing it to a simple integral, when the limits are expressed by the potential energy, and the other factor under the integral sign is a function of

* If V_q were infinite for finite values of ϵ_q, V would evidently be infinite for finite values of ϵ.

the potential energy, either alone or with quantities which are constant in the integration.

We may often avoid the inconvenience occasioned by formulae becoming illusory on account of discontinuities in the values of V_q as function of ϵ_q by substituting for the given discontinuous function a continuous function which is practically equivalent to the given function for the purposes of the evaluations desired. It only requires infinitesimal changes of potential energy to destroy the finite extensions-in-configuration of constant potential energy which are the cause of the difficulty.

In the case of an ensemble of systems canonically distributed in configuration, when V_q is, or may be regarded as, a continuous function of ϵ_q (within the limits considered), the probability that the potential energy of an unspecified system lies between the limits ϵ_q' and ϵ_q'' is given by the integral

$$\int_{\epsilon_q'}^{\epsilon_q''} e^{\frac{\psi_q - \epsilon_q}{\Theta} + \phi_q} d\epsilon_q, \tag{274}$$

where ψ may be determined by the condition that the value of the integral is unity, when the limits include all possible values of ϵ_q. In the same case, the average value in the ensemble of any function of the potential energy is given by the equation

$$\bar{u} = \int_{V_q=0}^{\epsilon_q=\infty} u e^{\frac{\psi - \epsilon_q}{\Theta} + \phi_q} d\epsilon_q. \tag{275}$$

When V_q is not a continuous function of ϵ_q, we may write dV_q for $e^{\phi_q} d\epsilon_q$ in these formulae.

In like manner also, for any given configuration, let us denote by V_p the extension-in-velocity below a certain limit of kinetic energy specified by ϵ_p. That is, let

$$V_p = \int \dots \int \Delta_p^{\frac{1}{2}} dp_1 \dots dp_n, \tag{276}$$

the integration being extended, with constant values of the coördinates, both internal and external, over all values of the momenta for which the kinetic energy is less than the limit ϵ_p. V_p will evidently be a continuous increasing function of ϵ_p which vanishes and becomes infinite with ϵ_p. Let us set

$$\phi_p = \log \frac{dV_p}{d\epsilon_p}. \tag{277}$$

The extension-in-velocity between any two limits of kinetic energy ϵ_p' and ϵ_p'' may be represented by the integral

$$\int_{\epsilon_p'}^{\epsilon_p''} e^{\phi_p} d\epsilon_p. \tag{278}$$

And in general, we may substitute $e^{\phi_p} d\epsilon_p$ for $\Delta_p^{\frac{1}{2}} dp_1 \ldots dp_n$ or $\Delta_q^{\frac{1}{2}} dq_1 \ldots dq_n$ in an n-fold integral in which the coördinates are constant, reducing it to a simple integral, when the limits are expressed by the kinetic energy, and the other factor under the integral sign is a function of the kinetic energy, either alone or with quantities which are constant in the integration.

It is easy to express V_p and ϕ_p in terms of ϵ_p. Since Δ_p is function of the coördinates alone, we have by definition

$$V_p = \Delta_p^{\frac{1}{2}} \int \ldots \int dp_1 \ldots dp_n \tag{279}$$

the limits of the integral being given by ϵ_p. That is, if

$$\epsilon_p = F(p_1, \ldots p_n), \tag{280}$$

the limits of the integral for $\epsilon_p = 1$, are given by the equation

$$F(p_1, \ldots p_n) = 1, \tag{281}$$

and the limits of the integral for $\epsilon_p = a^2$, are given by the equation

$$F(p_1, \ldots p_n) = a^2. \tag{282}$$

But since F represents a quadratic function, this equation may be written

$$F\left(\frac{p_1}{a}, \ldots \frac{p_n}{a}\right) = 1 \tag{283}$$

The value of V_p may also be put in the form

$$V_p = a^n \Delta_p^{\frac{1}{2}} \int \dots \int d\frac{p_1}{a} \dots d\frac{p_n}{a}. \tag{284}$$

Now we may determine V_p for $\epsilon_p = 1$ from (279) where the limits are expressed by (281), and V_p for $\epsilon_p = a^2$ from (284) taking the limits from (283). The two integrals thus determined are evidently identical, and we have

$$(V_p)_{\epsilon_p = a^2} = a^n (V_p)_{\epsilon_p = 1} \tag{285}$$

i. e., V_p varies as $\epsilon_p^{\frac{n}{2}}$. We may therefore set

$$V_p = C \epsilon_p^{\frac{n}{2}}, \qquad e^{\phi_p} = \frac{n}{2} C \epsilon_p^{\frac{n}{2}-1} \tag{286}$$

where C is a constant, at least for fixed values of the internal coördinates.

To determine this constant, let us consider the case of a canonical distribution, for which we have

$$\int_0^\infty e^{\frac{\psi_p - \epsilon_p}{\Theta} + \phi_p} d\epsilon_p = 1,$$

where

$$e^{\frac{\psi_p}{\Theta}} = (2\pi\Theta)^{-\frac{n}{2}}.$$

Substituting this value, and that of e^{ϕ_p} from (286), we get

$$\frac{n}{2} C \int_0^\infty e^{-\frac{\epsilon_p}{\Theta}} \epsilon_p^{\frac{n}{2}-1} d\epsilon_p = (2\pi\Theta)^{\frac{n}{2}},$$

$$\frac{n}{2} C \int_0^\infty e^{-\frac{\epsilon_p}{\Theta}} \left(\frac{\epsilon_p}{\Theta}\right)^{\frac{n}{2}-1} d\left(\frac{\epsilon_p}{\Theta}\right) = (2\pi)^{\frac{n}{2}},$$

$$\frac{n}{2} C \Gamma\left(\frac{n}{2}\right) = (2\pi)^{\frac{n}{2}}, \tag{287}$$

$$C = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{1}{2}n + 1)}.$$

Having thus determined the value of the constant C, we may

substitute it in the general expressions (286), and obtain the following values, which are perfectly general:

$$V_p = \frac{(2\pi\epsilon_p)^{\frac{n}{2}}}{\Gamma(\frac{1}{2}n+1)} \tag{288}$$

$$e^{\phi_p} = \frac{(2\pi)^{\frac{n}{2}}\epsilon_p^{\frac{n}{2}-1}}{\Gamma(\frac{1}{2}n)} \tag{289}$$

It will be observed that the values of V_p and ϕ_p for any given ϵ_p are independent of the configuration, and even of the nature of the system considered, except with respect to its number of degrees of freedom.

Returning to the canonical ensemble, we may express the probability that the kinetic energy of a system of a given configuration, but otherwise unspecified, falls within given limits, by either member of the following equation

$$\int e^{\frac{\psi_p-\epsilon_p}{\Theta}+\phi_p} d\epsilon_p = \frac{1}{\Gamma(\frac{1}{2}n)}\int e^{-\frac{\epsilon_p}{\Theta}}\left(\frac{\epsilon_p}{\Theta}\right)^{\frac{n}{2}-1} d\left(\frac{\epsilon_p}{\Theta}\right). \tag{290}$$

Since this value is independent of the coördinates it also represents the probability that the kinetic energy of an unspecified system of a canonical ensemble falls within the limits. The form of the last integral also shows that the probability that the ratio of the kinetic energy to the modulus

* Very similar values for V_q, e^{ϕ_q}, V, and e^{ϕ} may be found in the same way in the case discussed in the preceding foot-notes (see pages 54, 72, 77, and 79), in which ϵ_q is a quadratic function of the q's, and $\Delta_{\dot{q}}$ independent of the q's. In this case we have

$$V_q = \left(\frac{\Delta_{\dot{q}}}{\Delta_q}\right)^{\frac{1}{2}} \frac{(2\pi)^{\frac{n}{2}}(\epsilon_q-\epsilon_a)^{\frac{n}{2}}}{\Gamma(\frac{1}{2}n+1)},$$

$$e^{\phi_q} = \left(\frac{\Delta_{\dot{q}}}{\Delta_q}\right)^{\frac{1}{2}} \frac{(2\pi)^{\frac{n}{2}}(\epsilon_q-\epsilon_a)^{\frac{n}{2}-1}}{\Gamma(\frac{1}{2}n)},$$

$$V = \left(\frac{\Delta_{\dot{q}}}{\Delta_q}\right)^{\frac{1}{2}} \frac{(2\pi)^{n}(\epsilon-\epsilon_a)^{n}}{\Gamma(n+1)},$$

$$e^{\phi} = \left(\frac{\Delta_{\dot{q}}}{\Delta_q}\right)^{\frac{1}{2}} \frac{(2\pi)^{n}(\epsilon-\epsilon_a)^{n-1}}{\Gamma(n)}.$$

falls within given limits is independent also of the value of the modulus, being determined entirely by the number of degrees of freedom of the system and the limiting values of the ratio.

The average value of any function of the kinetic energy, either for the whole ensemble, or for any particular configuration, is given by

$$\bar{u} = \frac{1}{\Theta^{\frac{n}{2}}\Gamma(\frac{1}{2}n)} \int_0^\infty u\, e^{-\frac{\epsilon_p}{\Theta}} \epsilon_p^{\frac{n}{2}-1}\, d\epsilon_p \qquad *(291)$$

Thus:

$$\overline{\epsilon_p^{\,m}} = \frac{\Gamma(m+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}\Theta^m, \quad \text{if} \quad m+\tfrac{1}{2}n > 0; \qquad \dagger(292)$$

$$\overline{V_p} = \frac{\Gamma(n)}{\Gamma(\frac{1}{2}n+1)\,\Gamma(\frac{1}{2}n)}(2\pi\Theta)^{\frac{n}{2}}; \qquad (293)$$

* The corresponding equation for the average value of any function of the potential energy, when this is a quadratic function of the q's, and Δ_q is independent of the q's, is

$$\bar{u} = \frac{1}{\Theta^{\frac{n}{2}}\Gamma(\frac{1}{2}n)} \int_{\epsilon_a}^\infty u\, e^{-\frac{\epsilon_q-\epsilon_a}{\Theta}} (\epsilon_q-\epsilon_a)^{\frac{n}{2}-1}\, d\epsilon_q.$$

In the same case, the average value of any function of the (total) energy is given by the equation

$$\bar{u} = \frac{1}{\Theta^n\Gamma(n)} \int_0^\infty u\, e^{-\frac{\epsilon-\epsilon_a}{\Theta}} (\epsilon-\epsilon_a)^{n-1}\, d\epsilon.$$

Hence in this case

$$\overline{(\epsilon_q-\epsilon_a)^m} = \frac{\Gamma(m+\frac{1}{2}n)}{\Gamma(\frac{1}{2}n)}\Theta^m, \quad \text{if} \quad m+\frac{1}{2}n > 0.$$

$$\overline{(\epsilon-\epsilon_a)^m} = \frac{\Gamma(m+n)}{\Gamma(n)}\Theta^m, \quad \text{if} \quad m+n > 0.$$

$$\overline{e^{-\phi_q} V_q} = \overline{e^{-\phi} V} = \Theta,$$

$$\overline{\frac{d\phi_q}{d\epsilon_q}} = \frac{1}{\Theta}, \quad \text{if} \quad n > 2,$$

and

$$\overline{\frac{d\phi}{d\epsilon}} = \frac{1}{\Theta}, \quad \text{if} \quad n > 1.$$

If $n=1$, $e^{\phi} = 2\pi$ and $d\phi/d\epsilon = 0$ for any value of ϵ. If $n=2$, the case is the same with respect to ϕ_q.

† This equation has already been proved for positive integral powers of the kinetic energy. See page 77.

$$\overline{e^{\phi_p}} = \frac{\Gamma(n-1)}{[\Gamma(\frac{1}{2}n)]^2}(2\pi)^{\frac{n}{2}}\Theta^{\frac{n}{2}-1}, \qquad \text{if} \qquad n > 1; \tag{294}$$

$$\overline{\frac{d\phi_p}{d\epsilon_p}} = \frac{1}{\Theta}, \qquad \text{if} \qquad n > 2; \tag{295}$$

$$\overline{e^{-\phi_p}}\,\overline{V_p} = \Theta. \tag{296}$$

If $n = 2$, $\epsilon^{\phi_p} = 2\pi$, and $d\phi_p/d\epsilon_p = 0$, for any value of ϵ_p.

The definitions of V, V_q, and V_p give

$$V = \int\int dV_p\, dV_q \tag{297}$$

where the integrations cover all phases for which the energy is less than the limit ϵ, for which the value of V is sought. This gives

$$V = \int_{V_q=0}^{\epsilon_q=\epsilon} V_p\, dV_q, \tag{298}$$

and

$$e^{\phi} = \frac{dV}{d\epsilon} = \int_{V_q=0}^{\epsilon_q=\epsilon} e^{\phi_p}\, dV_q, \tag{299}$$

where V_p and ϵ^{ϕ_p} are connected with V_q by the equation

$$\epsilon_p + \epsilon_q = \text{constant} = \epsilon. \tag{300}$$

If $n > 2$, ϵ^{ϕ_p} vanishes at the upper limit, *i. e.*, for $\epsilon_p = 0$, and we get by another differentiation

$$e^{\phi}\frac{d\phi}{d\epsilon} = \int_{V_q=0}^{\epsilon_q=\epsilon} e^{\phi_p}\frac{d\phi_p}{d\epsilon_p}\, dV_q. \tag{301}$$

We may also write

$$V = \int_{V_q=0}^{\epsilon_q=\epsilon} V_p\, e^{\phi_q}\, d\epsilon_q, \tag{302}$$

$$e^{\phi} = \int_{V_q=0}^{\epsilon_q=\epsilon} e^{\phi_p+\phi_q}\, d\epsilon_q, \tag{303}$$

etc., when V_q is a continuous function of ϵ_q commencing with the value $V_q = 0$, or when we choose to attribute to V_q a fictitious continuity commencing with the value zero, as described on page 90.

If we substitute in these equations the values of V_p and ϵ^{ϕ_p} which we have found, we get

$$V = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{1}{2}n+1)} \int_{V_q=0}^{\epsilon_q=\epsilon} (\epsilon - \epsilon_q)^{\frac{n}{2}} \, dV_q, \tag{304}$$

$$e^{\phi} = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \int_{V_q=0}^{\epsilon_q=\epsilon} (\epsilon - \epsilon_q)^{\frac{n}{2}-1} \, dV_q, \tag{305}$$

where $e^{\phi_q} \, d\epsilon_q$ may be substituted for dV_q in the cases above described. If, therefore, n is known, and V_q as function of ϵ_q, V and ϵ^{ϕ} may be found by quadratures.

It appears from these equations that V is always a continuous increasing function of ϵ, commencing with the value $V = 0$, even when this is not the case with respect to V_q and ϵ_q. The same is true of e^{ϕ}, when $n > 2$, or when $n = 2$ if V_q increases continuously with ϵ_q from the value $V_q = 0$.

The last equation may be derived from the preceding by differentiation with respect to ϵ. Successive differentiations give, if $h < \frac{1}{2}n + 1$,

$$\frac{d^h V}{d\epsilon^h} = \int_{V_q=0}^{\epsilon_q=\epsilon} \frac{d^h V_p}{d\epsilon_p{}^h} \, dV_q = \frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{1}{2}n+1-h)} \int_{V_q=0}^{\epsilon_q=\epsilon} (\epsilon - \epsilon_q)^{\frac{n}{2}-h} \, dV_q. \tag{306}$$

$d^h V / d\epsilon^h$ is therefore positive if $h < \frac{1}{2}n + 1$. It is an increasing function of ϵ, if $h < \frac{1}{2}n$. If ϵ is not capable of being diminished without limit, $d^h V / d\epsilon^h$ vanishes for the least possible value of ϵ, if $h < \frac{1}{2}n$.

If n is even,

$$\frac{d^{\frac{n}{2}} V}{d\epsilon^{\frac{n}{2}}} = (2\pi)^{\frac{n}{2}} (V_q)_{\epsilon_q=\epsilon} \tag{307}$$

That is, V_q is the same function of ϵ_q, as $\frac{1}{(2\pi)^{\frac{n}{2}}} \frac{d^{\frac{n}{2}}V}{d\epsilon^{\frac{n}{2}}}$ of ϵ.

When n is large, approximate formulae will be more available. It will be sufficient to indicate the method proposed, without precise discussion of the limits of its applicability or of the degree of its approximation. For the value of e^{ϕ} corresponding to any given ϵ, we have

$$e^{\phi} = \int_{V_q=0}^{\epsilon_q=\epsilon} e^{\phi_p+\phi_q} d\epsilon_q = \int_0^{\epsilon} e^{\phi_p+\phi_q} d\epsilon_p, \tag{308}$$

where the variables are connected by the equation (300). The maximum value of $\phi_p + \phi_q$ is therefore characterized by the equation

$$\frac{d\phi_p}{d\epsilon_p} = \frac{d\phi_q}{d\epsilon_q}. \tag{309}$$

The values of ϵ_p and ϵ_q determined by this maximum we shall distinguish by accents, and mark the corresponding values of functions of ϵ_p and ϵ_q in the same way. Now we have by Taylor's theorem

$$\phi_p = \phi_p' + \left(\frac{d\phi_p}{d\epsilon_p}\right)'(\epsilon_p - \epsilon_p') + \left(\frac{d^2\phi_p}{d\epsilon_p^{\,2}}\right)'\frac{(\epsilon_p - \epsilon_p')^2}{2} + \text{etc.} \tag{310}$$

$$\phi_q = \phi_q' + \left(\frac{d\phi_q}{d\epsilon_q}\right)'(\epsilon_q - \epsilon_q') + \left(\frac{d^2\phi_q}{d\epsilon_q^{\,2}}\right)'\frac{(\epsilon_q - \epsilon_q')}{2} + \text{etc.} \tag{311}$$

If the approximation is sufficient without going beyond the quadratic terms, since by (300)

$$\epsilon_p - \epsilon_p' = -(\epsilon_q - \epsilon_q'),$$

we may write

$$e^{\phi} = e^{\phi_p'+\phi_q'} \int_{-\infty}^{+\infty} e^{\left[\left(\frac{d^2\phi_p}{d\epsilon_p^{\,2}}\right)' + \left(\frac{d^2\phi_q}{d\epsilon_q^{\,2}}\right)'\right]\frac{(\epsilon_q-\epsilon_q')^2}{2}} d\epsilon_q, \tag{312}$$

where the limits have been made $\pm\infty$ for analytical simplicity. This is allowable when the quantity in the square brackets has a very large negative value, since the part of the integral

corresponding to other than very small values of $\epsilon_q - \epsilon_q'$ may be regarded as a vanishing quantity.

This gives

$$e^{\phi} = e^{\phi_p' + \phi_q'} \left[\frac{-2\pi}{\left(\frac{d^2\phi_p}{d\epsilon_p^2}\right)' + \left(\frac{d^2\phi_q}{d\epsilon_q^2}\right)'} \right]^{\frac{1}{2}} \tag{313}$$

or

$$\phi = \phi_p' + \phi_q' + \tfrac{1}{2}\log(2\pi) - \tfrac{1}{2}\log\left[-\left(\frac{d^2\phi_p}{d\epsilon_p^2}\right)' - \left(\frac{d^2\phi_q}{d\epsilon_q^2}\right)' \right]. \tag{314}$$

From this equation, with (289), (300) and (309), we may determine the value of ϕ corresponding to any given value of ϵ, when ϕ_q is known as function of ϵ_q.

Any two systems may be regarded as together forming a third system. If we have V or ϕ given as function of ϵ for any two systems, we may express by quadratures V and ϕ for the system formed by combining the two. If we distinguish by the suffixes $(\ )_1$, $(\ )_2$, $(\ )_{12}$ the quantities relating to the three systems, we have easily from the definitions of these quantities

$$V_{12} = \int\int dV_1 dV_2 = \int V_2 dV_1 = \int V_1 dV_2 = \int V_1 e^{\phi_2} d\epsilon_2, \tag{315}$$

$$e^{\phi_{12}} = \int e^{\phi_2} dV_1 = \int e^{\phi_1} dV_2 = \int e^{\phi_1 + \phi_2} d\epsilon_2, \tag{316}$$

where the double integral is to be taken within the limits

$$V_1 = 0,\ V_2 = 0, \text{ and } \epsilon_1 + \epsilon_2 = \epsilon_{12},$$

and the variables in the single integrals are connected by the last of these equations, while the limits are given by the first two, which characterize the least possible values of ϵ_1 and ϵ_2 respectively.

It will be observed that these equations are identical in form with those by which V and ϕ are derived from V_p or ϕ_p and V_q or ϕ_q, except that they do not admit in the general case those transformations which result from substituting for V_p or ϕ_p the particular functions which these symbols always represent.

Similar formulae may be used to derive V_q or ϕ_q for the compound system, when one of these quantities is known as function of the potential energy in each of the systems combined.

The operation represented by such an equation as

$$e^{\phi_{12}} = \int e^{\phi_1} e^{\phi_2} d\epsilon_1$$

is identical with one of the fundamental operations of the theory of errors, viz., that of finding the probability of an error from the probabilities of partial errors of which it is made up. It admits a simple geometrical illustration.

We may take a horizontal line as an axis of abscissas, and lay off ϵ_1 as an abscissa measured to the right of any origin, and erect e^{ϕ_1} as a corresponding ordinate, thus determining a certain curve. Again, taking a different origin, we may lay off ϵ_2 as abscissas measured to the left, and determine a second curve by erecting the ordinates e^{ϕ_2}. We may suppose the distance between the origins to be ϵ_{12}, the second origin being to the right if ϵ_{12} is positive. We may determine a third curve by erecting at every point in the line (between the least values of ϵ_1 and ϵ_2) an ordinate which represents the product of the two ordinates belonging to the curves already described. The area between this third curve and the axis of abscissas will represent the value of $e^{\phi_{12}}$. To get the value of this quantity for varying values of ϵ_{12}, we may suppose the first two curves to be rigidly constructed, and to be capable of being moved independently. We may increase or diminish ϵ_{12} by moving one of these curves to the right or left. The third curve must be constructed anew for each different value of ϵ_{12}.

CHAPTER IX.

THE FUNCTION ϕ AND THE CANONICAL DISTRIBUTION.

In this chapter we shall return to the consideration of the canonical distribution, in order to investigate those properties which are especially related to the function of the energy which we have denoted by ϕ.

If we denote by N, as usual, the total number of systems in the ensemble,

$$Ne^{\frac{\psi-\epsilon}{\Theta}+\phi}d\epsilon$$

will represent the number having energies between the limits ϵ and $\epsilon + d\epsilon$. The expression

$$Ne^{\frac{\psi-\epsilon}{\Theta}+\phi} \tag{317}$$

represents what may be called the *density-in-energy*. This vanishes for $\epsilon = \infty$, for otherwise the necessary equation

$$\int_{V=0}^{\epsilon=\infty} e^{\frac{\psi-\epsilon}{\Theta}+\phi}d\epsilon = 1 \tag{318}$$

could not be fulfilled. For the same reason the density-in-energy will vanish for $\epsilon = -\infty$, if that is a possible value of the energy. Generally, however, the least possible value of the energy will be a finite value, for which, if $n > 2$, e^{ϕ} will vanish,* and therefore the density-in-energy. Now the density-in-energy is necessarily positive, and since it vanishes for extreme values of the energy if $n > 2$, it must have a maximum in such cases, in which the energy may be said to have

* See page 96.

its most common or most probable value, and which is determined by the equation

$$\frac{d\phi}{d\epsilon} = \frac{1}{\Theta}. \tag{319}$$

This value of $d\phi/d\epsilon$ is also, when $n > 2$, its average value in the ensemble. For we have identically, by integration by parts,

$$\int_{V=0}^{\epsilon=\infty} \frac{d\phi}{d\epsilon} e^{\frac{\psi-\epsilon}{\Theta}+\phi} d\epsilon = \left[e^{\frac{\psi-\epsilon}{\Theta}+\phi} \right]_{V=0}^{\epsilon=\infty} + \frac{1}{\Theta} \int_{V=0}^{\epsilon=\infty} e^{\frac{\psi-\epsilon}{\Theta}+\phi} d\epsilon. \tag{320}$$

If $n > 2$, the expression in the brackets, which multiplied by N would represent the density-in-energy, vanishes at the limits, and we have by (269) and (318)

$$\overline{\frac{d\phi}{d\epsilon}} = \frac{1}{\Theta}. \tag{321}$$

It appears, therefore, that for systems of more than two degrees of freedom, the average value of $d\phi/d\epsilon$ in an ensemble canonically distributed is identical with the value of the same differential coefficient as calculated for the most common energy in the ensemble, both values being reciprocals of the modulus.

Hitherto, in our consideration of the quantities V, V_q, V_p, ϕ, ϕ_q, ϕ_p, we have regarded the external coördinates as constant. It is evident, however, from their definitions that V and ϕ are in general functions of the external coördinates and the energy (ϵ), that V_q and ϕ_q are in general functions of the external coördinates and the potential energy (ϵ_q). V_p and ϕ_p we have found to be functions of the kinetic energy (ϵ_p) alone. In the equation

$$e^{-\frac{\psi}{\Theta}} = \int_{V=0}^{\epsilon=\infty} e^{-\frac{\epsilon}{\Theta}+\phi} d\epsilon, \tag{322}$$

by which ψ may be determined, Θ and the external coördinates (contained implicitly in ϕ) are constant in the integration. The equation shows that ψ is a function of these constants.

If their values are varied, we shall have by differentiation, if $n > 2$,

$$e^{-\frac{\psi}{\Theta}}\left(-\frac{1}{\Theta}\,d\psi + \frac{\psi}{\Theta^2}\,d\Theta\right) = \frac{1}{\Theta^2}\,d\Theta \int_{V=0}^{\epsilon=\infty} \epsilon e^{-\frac{\epsilon}{\Theta}+\phi}\,d\epsilon$$

$$+ da_1 \int_{V=0}^{\epsilon=\infty} \frac{d\phi}{da_1}\, e^{-\frac{\epsilon}{\Theta}+\phi}\,d\epsilon + da_2 \int_{V=0}^{\epsilon=\infty} \frac{d\phi}{da_2}\, e^{-\frac{\epsilon}{\Theta}+\phi}\,d\epsilon + \text{etc.} \quad (323)$$

(Since e^{ϕ} vanishes with V, when $n > 2$, there are no terms due to the variations of the limits.) Hence by (269)

$$-\frac{1}{\Theta}\,d\psi + \frac{\psi}{\Theta^2}\,d\Theta = \frac{\bar{\epsilon}}{\Theta^2}\,d\Theta + \overline{\frac{d\phi}{da_1}}\,da_1 + \overline{\frac{d\phi}{da_2}}\,da_2 + \text{etc.}, \quad (324)$$

or, since

$$\frac{\psi + \bar{\epsilon}}{\Theta} = \bar{\eta}, \quad (325)$$

$$d\psi = \bar{\eta}\,d\Theta - \Theta\overline{\frac{d\phi}{da_1}}\,da_1 - \Theta\overline{\frac{d\phi}{da_2}}\,da_2 - \text{etc.} \quad (326)$$

Comparing this with (112), we get

$$\overline{\frac{d\phi}{da_1}} = \frac{A_1}{\Theta}, \quad \overline{\frac{d\phi}{da_2}} = \frac{A_2}{\Theta}, \quad \text{etc.} \quad (327)$$

The first of these equations might be written*

$$\overline{\left(\frac{d\phi}{da_1}\right)}_{\epsilon,a} = -\overline{\left(\frac{d\phi}{d\epsilon}\right)}_{a}\left(\overline{\frac{d\epsilon}{da_1}}\right)_{a,q} \quad (328)$$

but must not be confounded with the equation

$$\overline{\left(\frac{d\phi}{da_1}\right)}_{\epsilon,a} = -\overline{\left(\frac{d\phi}{d\epsilon}\right)_{a}\left(\frac{d\epsilon}{da_1}\right)_{\phi,a}} \quad (329)$$

which is derived immediately from the identity

$$\left(\frac{d\phi}{da_1}\right)_{\epsilon,a} = -\left(\frac{d\phi}{d\epsilon}\right)_{a}\left(\frac{d\epsilon}{da_1}\right)_{\phi,a} \quad (330)$$

* See equations (321) and (104). Suffixes are here added to the differential coefficients, to make the meaning perfectly distinct, although the same quantities may be written elsewhere without the suffixes, when it is believed that there is no danger of misapprehension. The suffixes indicate the quantities which are constant in the differentiation, the single letter a standing for all the letters a_1, a_2, etc., or all except the one which is explicitly varied.

Moreover, if we eliminate $d\psi$ from (326) by the equation

$$d\psi = \Theta d\bar{\eta} + \bar{\eta} d\Theta + d\bar{\epsilon}, \tag{331}$$

obtained by differentiating (325), we get

$$d\bar{\epsilon} = -\Theta d\bar{\eta} - \Theta \overline{\frac{d\phi}{da_1}} da_1 - \Theta \overline{\frac{d\phi}{da_2}} da_2 - \text{etc.}, \tag{332}$$

or by (321),

$$-d\bar{\eta} = \overline{\frac{d\phi}{d\epsilon}} d\bar{\epsilon} + \overline{\frac{d\phi}{da_1}} da_1 + \overline{\frac{d\phi}{da_2}} da_2 + \text{etc.} \tag{333}$$

Except for the signs of average, the second member of this equation is the same as that of the identity

$$d\phi = \frac{d\phi}{d\epsilon} d\epsilon + \frac{d\phi}{da_1} da_1 + \frac{d\phi}{da_2} da_2 + \text{etc.} \tag{334}$$

For the more precise comparison of these equations, we may suppose that the energy in the last equation is some definite and fairly representative energy in the ensemble. For this purpose we might choose the average energy. It will perhaps be more convenient to choose the most common energy, which we shall denote by ϵ_0. The same suffix will be applied to functions of the energy determined for this value. Our identity then becomes

$$d\phi_0 = \left(\frac{d\phi}{d\epsilon}\right)_0 d\epsilon_0 + \left(\frac{d\phi}{da_1}\right)_0 da_1 + \left(\frac{d\phi}{da_2}\right)_0 da_2 + \text{etc.} \tag{335}$$

It has been shown that

$$\overline{\frac{d\phi}{d\epsilon}} = \left(\frac{d\phi}{d\epsilon}\right)_0 = \frac{1}{\Theta}, \tag{336}$$

when $n > 2$. Moreover, since the external coördinates have constant values throughout the ensemble, the values of $d\phi/da_1$, $d\phi/da_2$, etc. vary in the ensemble only on account of the variations of the energy (ϵ), which, as we have seen, may be regarded as sensibly constant throughout the ensemble, when n is very great. In this case, therefore, we may regard the average values

$$\overline{\frac{d\phi}{da_1}}, \quad \overline{\frac{d\phi}{da_2}}, \quad \text{etc.},$$

as practically equivalent to the values relating to the most common energy

$$\left(\frac{d\phi}{da_1}\right)_0, \quad \left(\frac{d\phi}{da_2}\right)_0, \quad \text{etc.}$$

In this case also $d\bar{\epsilon}$ is practically equivalent to $d\epsilon_0$. We have therefore, for very large values of n,

$$-d\bar{\eta} = d\phi_0 \tag{337}$$

approximately. That is, except for an additive constant, $-\bar{\eta}$ may be regarded as practically equivalent to ϕ_0, when the number of degrees of freedom of the system is very great. It is not meant by this that the variable part of $\bar{\eta} + \phi_0$ is numerically of a lower order of magnitude than unity. For when n is very great, $-\bar{\eta}$ and ϕ_0 are very great, and we can only conclude that the variable part of $\bar{\eta} + \phi_0$ is insignificant compared with the variable part of $\bar{\eta}$ or of ϕ_0, taken separately.

Now we have already noticed a certain correspondence between the quantities Θ and $\bar{\eta}$ and those which in thermodynamics are called temperature and entropy. The property just demonstrated, with those expressed by equation (336), therefore suggests that the quantities ϕ and $d\epsilon/d\phi$ may also correspond to the thermodynamic notions of entropy and temperature. We leave the discussion of this point to a subsequent chapter, and only mention it here to justify the somewhat detailed investigation of the relations of these quantities.

We may get a clearer view of the limiting form of the relations when the number of degrees of freedom is indefinitely increased, if we expand the function ϕ in a series arranged according to ascending powers of $\epsilon - \epsilon_0$. This expansion may be written

$$\phi = \phi_0 + \left(\frac{d\phi}{d\epsilon}\right)_0 (\epsilon - \epsilon_0) + \left(\frac{d^2\phi}{d\epsilon^2}\right)_0 \frac{(\epsilon - \epsilon_0)^2}{2} + \left(\frac{d^3\phi}{d\epsilon^3}\right)_0 \frac{(\epsilon - \epsilon_0)^3}{\underline{|3}} + \text{etc.} \tag{338}$$

Adding the identical equation

$$\frac{\psi-\epsilon}{\Theta}=\frac{\psi-\epsilon_0}{\Theta}-\frac{\epsilon-\epsilon_0}{\Theta},$$

we get by (336)

$$\frac{\psi-\epsilon}{\Theta}+\phi=\frac{\psi-\epsilon_0}{\Theta}+\phi_0+\left(\frac{d^2\phi}{d\epsilon^2}\right)\frac{(\epsilon-\epsilon_0)^2}{2}+\left(\frac{d^3\phi}{d\epsilon^3}\right)\frac{(\epsilon-\epsilon_0)^3}{\lfloor 3}+\text{etc.} \tag{339}$$

Substituting this value in

$$\int_{\epsilon'}^{\epsilon''} e^{\frac{\psi-\epsilon}{\Theta}+\phi}\,d\epsilon,$$

which expresses the probability that the energy of an unspecified system of the ensemble lies between the limits ϵ' and ϵ'', we get

$$e^{\frac{\psi-\epsilon_0}{\Theta}+\phi_0}\int_{\epsilon'}^{\epsilon''} e^{\left(\frac{d^2\phi}{d\epsilon^2}\right)_0\frac{(\epsilon-\epsilon_0)^2}{2}+\left(\frac{d^3\phi}{d\epsilon^3}\right)_0\frac{(\epsilon-\epsilon_0)^3}{\lfloor 3}+\text{etc.}}\,d\epsilon. \tag{340}$$

When the number of degrees of freedom is very great, and $\epsilon-\epsilon_0$ in consequence very small, we may neglect the higher powers and write*

$$e^{\frac{\psi-\epsilon_0}{\Theta}+\phi_0}\int_{\epsilon'}^{\epsilon''} e^{\left(\frac{d^2\phi}{d\epsilon^2}\right)_0\frac{(\epsilon-\epsilon_0)^2}{2}}\,d\epsilon. \tag{341}$$

This shows that for a very great number of degrees of freedom the probability of deviations of energy from the most probable value (ϵ_0) approaches the form expressed by the 'law of errors.' With this approximate law, we get

* If a higher degree of accuracy is desired than is afforded by this formula, it may be multiplied by the series obtained from

$$e^{\left(\frac{d^3\phi}{d\epsilon^3}\right)_0\frac{(\epsilon-\epsilon_0)^3}{\lfloor 3}+\text{etc.}}$$

by the ordinary formula for the expansion in series of an exponential function. There would be no especial analytical difficulty in taking account of a moderate number of terms of such a series, which would commence

$$1+\left(\frac{d^3\phi}{d\epsilon^3}\right)_0\frac{(\epsilon-\epsilon_0)^3}{\lfloor 3}+\left(\frac{d^4\phi}{d\epsilon^4}\right)_0\frac{(\epsilon-\epsilon_0)^4}{\lfloor 4}+\text{etc.}$$

$$e^{\frac{\psi-\epsilon_0}{\Theta}+\phi_0}\left(\frac{-2\pi}{\left(\frac{d^2\phi}{d\epsilon^2}\right)_0}\right)^{\frac{1}{2}}=1, \tag{342}$$

$$\bar{\epsilon}=\epsilon_0, \qquad \overline{(\epsilon-\epsilon_0)^2}=-\left(\frac{d^2\phi}{d\epsilon^2}\right)^{-}, \tag{343}$$

whence

$$\frac{\psi-\epsilon_0}{\Theta}+\phi_0=\tfrac{1}{2}\log\frac{\left(\frac{d^2\phi}{d\epsilon^2}\right)_0}{-2\pi}=-\tfrac{1}{2}\log\left(2\pi\overline{(\epsilon-\bar{\epsilon})^2}\right). \tag{344}$$

Now it has been proved in Chapter VII that

$$\overline{(\epsilon-\bar{\epsilon})^2}=\frac{2}{n}\frac{d\bar{\epsilon}}{d\bar{\epsilon}_p}\bar{\epsilon}_p^{\,2}.$$

We have therefore

$$\bar{\eta}+\phi_0=\frac{\psi-\bar{\epsilon}}{\Theta}+\phi_0=-\tfrac{1}{2}\log\left(2\pi\overline{(\epsilon-\bar{\epsilon})^2}\right)=-\tfrac{1}{2}\log\left(\frac{4\pi}{n}\frac{d\bar{\epsilon}}{d\bar{\epsilon}_p}\bar{\epsilon}_p^{\,2}\right) \tag{345}$$

approximately. The order of magnitude of $\bar{\eta}-\phi_0$ is therefore that of $\log n$. This magnitude is mainly constant. The order of magnitude of $\bar{\eta}+\phi_0-\frac{1}{2}\log n$ is that of unity. The order of magnitude of ϕ_0, and therefore of $-\bar{\eta}$, is that of n.*

Equation (338) gives for the first approximation

$$\bar{\phi}-\phi_0=\left(\frac{d^2\phi}{d\epsilon^2}\right)_0\frac{\overline{(\epsilon-\epsilon_0)^2}}{2}=-\tfrac{1}{2}, \tag{346}$$

$$\overline{(\phi-\phi_0)(\epsilon-\epsilon_0)}=\frac{\overline{(\epsilon-\epsilon_0)^2}}{\Theta}=\frac{d\bar{\epsilon}}{d\bar{\epsilon}_p}\bar{\epsilon}_p, \tag{347}$$

$$\overline{(\phi-\phi_0)^2}=\frac{\overline{(\epsilon-\epsilon_0)^2}}{\Theta^2}=\frac{n}{2}\frac{d\bar{\epsilon}}{d\bar{\epsilon}_p} \tag{348}$$

The members of the last equation have the order of magnitude of n. Equation (338) gives also for the first approximation

$$\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}=\left(\frac{d^2\phi}{d\epsilon^2}\right)_0(\epsilon-\epsilon_0),$$

* Compare (289), (314).

whence

$$\overline{\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)(\epsilon-\epsilon_0)}=\left(\frac{d^2\phi}{d\epsilon^2}\right)_0\overline{(\epsilon-\epsilon_0)^2}=-1, \qquad (349)$$

$$\overline{\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)^2}=\left(\frac{d^2\phi}{d\epsilon^2}\right)_0^2\overline{(\epsilon-\epsilon_0)^2}=\frac{1}{\overline{(\epsilon-\epsilon_0)^2}}=-\left(\frac{d^2\phi}{d\epsilon^2}\right)_0. \qquad (350)$$

This is of the order of magnitude of n.*

It should be observed that the approximate distribution of the ensemble in energy according to the 'law of errors' is not dependent on the particular form of the function of the energy which we have assumed for the index of probability (η). In any case, we must have

$$\int_{V=0}^{\epsilon=\infty} e^{\eta+\phi}\,d\epsilon=1, \qquad (351)$$

where $e^{\eta+\phi}$ is necessarily positive. This requires that it shall vanish for $\epsilon=\infty$, and also for $\epsilon=-\infty$, if this is a possible value. It has been shown in the last chapter that if ϵ has a (finite) least possible value (which is the usual case) and $n>2$, e^{ϕ} will vanish for that least value of ϵ. In general therefore $\eta+\phi$ will have a maximum, which determines the most probable value of the energy. If we denote this value by ϵ_0, and distinguish the corresponding values of the functions of the energy by the same suffix, we shall have

$$\left(\frac{d\eta}{d\epsilon}\right)_0+\left(\frac{d\phi}{d\epsilon}\right)_0=0. \qquad (352)$$

The probability that an unspecified system of the ensemble

* We shall find hereafter that the equation

$$\overline{\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)(\epsilon-\bar{\epsilon})}=-1$$

is exact for any value of n greater than 2, and that the equation

$$\overline{\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)^2}=-\frac{d^2\phi}{d\epsilon^2}$$

is exact for any value of n greater than 4.

falls within any given limits of energy (ϵ' and ϵ'') is represented by

$$\int_{\epsilon'}^{\epsilon''} e^{\eta+\phi}\, d\epsilon.$$

If we expand η and ϕ in ascending powers of $\epsilon - \epsilon_0$, without going beyond the squares, the probability that the energy falls within the given limits takes the form of the 'law of errors' —

$$e^{\phi_0+\eta_0} \int_{\epsilon'}^{\epsilon''} e^{\left[\left(\frac{d^2\eta}{d\epsilon^2}\right)_0+\left(\frac{d^2\phi}{d\epsilon^2}\right)_0\right]\frac{(\epsilon-\epsilon_0)^2}{2}} d\epsilon. \tag{353}$$

This gives

$$\eta_0 + \phi_0 = \tfrac{1}{2} \log\left[\frac{-1}{2\pi}\left(\frac{d^2\eta}{d\epsilon_2{}^2}\right)_0 - \frac{1}{2\pi}\left(\frac{d^2\phi}{d\epsilon^2}\right)_0\right], \tag{354}$$

and

$$\overline{(\epsilon - \epsilon_0)^2} = \left[-\left(\frac{d^2\eta}{d\epsilon^2}\right)_0 - \left(\frac{d^2\phi}{d\epsilon^2}\right)_0\right]^{-1}. \tag{355}$$

We shall have a close approximation in general when the quantities equated in (355) are very small, *i. e.*, when

$$-\left(\frac{d^2\eta}{d\epsilon^2}\right)_0 - \left(\frac{d^2\phi}{d\epsilon^2}\right)_0 \tag{356}$$

is very great. Now when n is very great, $-d^2\phi/d\epsilon^2$ is of the same order of magnitude, and the condition that (356) shall be very great does not restrict very much the nature of the function η.

We may obtain other properties pertaining to average values in a canonical ensemble by the method used for the average of $d\phi/d\epsilon$. Let u be any function of the energy, either alone or with Θ and the external coördinates. The average value of u in the ensemble is determined by the equation

$$\bar{u} = \int_{V=0}^{\epsilon=\infty} u\, e^{\frac{\psi-\epsilon}{\Theta}+\phi}\, d\epsilon. \tag{357}$$

Now we have identically

$$\int_{V=0}^{\epsilon=\infty}\left(\frac{du}{d\epsilon}-\frac{u}{\Theta}+u\frac{d\phi}{d\epsilon}\right)e^{\frac{\psi-\epsilon}{\Theta}+\phi}\,d\epsilon=\left[u\,e^{\frac{\psi-\epsilon}{\Theta}+\phi}\right]_{V=0}^{\epsilon=\infty} \tag{358}$$

Therefore, by the preceding equation

$$\overline{\frac{du}{d\epsilon}}-\frac{\bar{u}}{\Theta}+\overline{u\frac{d\phi}{d\epsilon}}=\left[u\,e^{\frac{\psi-\epsilon}{\Theta}+\phi}\right]_{V=0}^{\epsilon=\infty} \tag{359}$$

If we set $u=1$, (a value which need not be excluded,) the second member of this equation vanishes, as shown on page 101, if $n>2$, and we get

$$\overline{\frac{d\phi}{d\epsilon}}=\frac{1}{\Theta}, \tag{360}$$

as before. It is evident from the same considerations that the second member of (359) will always vanish if $n>2$, unless u becomes infinite at one of the limits, in which case a more careful examination of the value of the expression will be necessary. To facilitate the discussion of such cases, it will be convenient to introduce a certain limitation in regard to the nature of the system considered. We have necessarily supposed, in all our treatment of systems canonically distributed, that the system considered was such as to be capable of the canonical distribution with the given value of the modulus. We shall now suppose that the system is such as to be capable of a canonical distribution with any (finite)† modulus. Let us see what cases we exclude by this last limitation.

* A more general equation, which is not limited to ensembles canonically distributed, is

$$\overline{\frac{du}{d\epsilon}}+\overline{u\frac{d\eta}{d\epsilon}}+\overline{u\frac{d\phi}{d\epsilon}}=\left[u\,e^{\eta+\phi}\right]_{V=0}^{\epsilon=\infty}$$

where η denotes, as usual, the index of probability of phase.

† The term *finite* applied to the modulus is intended to exclude the value zero as well as infinity.

The impossibility of a canonical distribution occurs when the equation

$$e^{-\frac{\psi}{\Theta}} = \int_{V=0}^{\epsilon=\infty} e^{-\frac{\epsilon}{\Theta}+\phi}\, d\epsilon \qquad (361)$$

fails to determine a finite value for ψ. Evidently the equation cannot make ψ an infinite positive quantity, the impossibility therefore occurs when the equation makes $\psi = -\infty$. Now we get easily from (191)

$$d\frac{\psi}{\Theta} = -\frac{\bar{\epsilon}}{\Theta^2}\, d\Theta.$$

If the canonical distribution is possible for any values of Θ, we can apply this equation so long as the canonical distribution is possible. The equation shows that as Θ is increased (without becoming infinite) $-\psi$ cannot become infinite unless $\bar{\epsilon}$ simultaneously becomes infinite, and that as Θ is decreased (without becoming zero) $-\psi$ cannot become infinite unless simultaneously $\bar{\epsilon}$ becomes an infinite negative quantity. The corresponding cases in thermodynamics would be bodies which could absorb or give out an infinite amount of heat without passing certain limits of temperature, when no external work is done in the positive or negative sense. Such infinite values present no analytical difficulties, and do not contradict the general laws of mechanics or of thermodynamics, but they are quite foreign to our ordinary experience of nature. In excluding such cases (which are certainly not entirely devoid of interest) we do not exclude any which are analogous to any actual cases in thermodynamics.

We assume then that for any finite value of Θ the second member of (361) has a finite value.

When this condition is fulfilled, the second member of (359) will vanish for $u = e^{-\phi} V$. For, if we set $\Theta' = 2\Theta$,

$$e^{-\frac{\epsilon}{\Theta}} V = e^{-\frac{\epsilon}{\Theta}} \int_{V=0}^{\epsilon} e^{\phi}\, d\epsilon \leqq e^{-\frac{\epsilon}{\Theta'}} \int_{V=0}^{\epsilon} e^{-\frac{\epsilon}{\Theta'}+\phi}\, d\epsilon \leqq e^{-\frac{\epsilon'}{\Theta'}} e^{-\frac{\psi'}{\Theta'}},$$

where ψ' denotes the value of ψ for the modulus Θ'. Since the last member of this formula vanishes for $\epsilon = \infty$, the less value represented by the first member must also vanish for the same value of ϵ. Therefore the second member of (359), which differs only by a constant factor, vanishes at the upper limit. The case of the lower limit remains to be considered. Now

$$e^{-\frac{\epsilon}{\Theta}} V \leqq \int_{V=0}^{\epsilon} e^{-\frac{\epsilon}{\Theta}+\phi}\, d\epsilon.$$

The second member of this formula evidently vanishes for the value of ϵ, which gives $V = 0$, whether this be finite or negative infinity. Therefore, the second member of (359) vanishes at the lower limit also, and we have

$$1 - \overline{e^{-\phi} V \frac{d\phi}{d\epsilon}} - \overline{e^{-\phi}} \frac{\overline{V}}{\Theta} + \overline{e^{-\phi} V \frac{d\phi}{d\epsilon}} = 0,$$

or

$$\overline{e^{-\phi}} \overline{V} = \Theta. \tag{362}$$

This equation, which is subject to no restriction in regard to the value of n, suggests a connection or analogy between the function of the energy of a system which is represented by $\epsilon^{-\phi} V$ and the notion of temperature in thermodynamics. We shall return to this subject in Chapter XIV.

If $n > 2$, the second member of (359) may easily be shown to vanish for any of the following values of u viz.: ϕ, e^{ϕ}, ϵ, ϵ^m, where m denotes any positive number. It will also vanish, when $n > 4$, for $u = d\phi/d\epsilon$, and when $n > 2h$ for $u = e^{-\phi}\, d^h V/d\epsilon^h$. When the second member of (359) vanishes, and $n > 2$, we may write

$$\overline{(u - \bar{u})\left(\frac{d\phi}{d\epsilon} - \frac{1}{\Theta}\right)} = \overline{u \frac{d\phi}{d\epsilon}} - \frac{\bar{u}}{\Theta} = -\overline{\frac{du}{d\epsilon}}. \tag{363}$$

We thus obtain the following equations:

If $n > 2$,

$$\overline{(\phi - \bar{\phi})\left(\frac{d\phi}{d\epsilon} - \frac{1}{\Theta}\right)} = \overline{\phi \frac{d\phi}{d\epsilon}} - \frac{\bar{\phi}}{\Theta} = -\frac{1}{\Theta}, \tag{364}$$

$$\overline{(e^{\phi}-\overline{e^{\phi}})\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)}=\overline{e^{\phi}\frac{d\phi}{d\epsilon}}-\frac{\overline{e^{\phi}}}{\Theta}=-\overline{e^{\phi}\frac{d\phi}{d\epsilon}}, \qquad (365)$$

or

$$2\overline{e^{\phi}\frac{d\phi}{d\epsilon}}=\overline{e^{\phi}}\,\overline{\frac{d\phi}{d\epsilon}}=\frac{\overline{e^{\phi}}}{\Theta}, \qquad (366)$$

$$\overline{(\epsilon-\bar{\epsilon})\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)}=\overline{\epsilon\frac{d\phi}{d\epsilon}}-\frac{\bar{\epsilon}}{\Theta}=-1, \qquad *(367)$$

$$\overline{(\epsilon^{m}-\overline{\epsilon^{m}})\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)}=\overline{\epsilon^{m}\frac{d\phi}{d\epsilon}}-\frac{\overline{\epsilon^{m}}}{\Theta}=-m\overline{\epsilon^{m-1}}. \qquad (368)$$

If $n > 4$,

$$\overline{\left(\frac{d\phi}{d\epsilon}-\frac{1}{\Theta}\right)^{2}}=\overline{\left(\frac{d\phi}{d\epsilon}\right)^{2}}-\frac{1}{\Theta^{2}}=-\overline{\left(\frac{d^{2}\phi}{d\epsilon^{2}}\right)^{2}}. \qquad \dagger(369)$$

If $n > 2h$,

$$\overline{e^{-\phi}\frac{d^{h}V}{d\epsilon^{h}}\frac{d\phi}{d\epsilon}}-\frac{1}{\Theta}\overline{e^{-\phi}\frac{d^{h}V}{d\epsilon^{h}}}=\overline{e^{-\phi}\frac{d^{h}V}{d\epsilon^{h}}\frac{d\phi}{d\epsilon}}-\overline{e^{-\phi}\frac{d^{h+1}V}{d\epsilon^{h+1}}}$$

or

$$\overline{e^{-\phi}\frac{d^{h+1}V}{d\epsilon^{h+1}}}=\frac{1}{\Theta}\overline{e^{-\phi}\frac{d^{h}V}{d\epsilon^{h}}} \qquad (370)$$

whence

$$\overline{e^{-\phi}\frac{d^{h+1}V}{d\epsilon^{h+1}}}=\frac{1}{\Theta^{h}}. \qquad (371)$$

Giving h the values 1, 2, 3, etc., we have

$$\overline{\frac{d\phi}{d\epsilon}}=\frac{1}{\Theta}, \quad \text{if } n>2,$$

$$\overline{\frac{d^{2}\phi}{d\epsilon^{2}}}+\overline{\left(\frac{d\phi}{d\epsilon}\right)^{2}}=\frac{1}{\Theta^{2}} \quad \text{if } n>4,$$

as already obtained. Also

$$\overline{\frac{d^{3}\phi}{d\epsilon^{3}}}+3\overline{\frac{d^{2}\phi}{d\epsilon^{2}}\frac{d\phi}{d\epsilon}}+\overline{\left(\frac{d\phi}{d\epsilon}\right)^{3}}=\frac{1}{\Theta^{3}} \quad \text{if } n>6. \qquad (372)$$

* This equation may also be obtained from equations (252) and (321). Compare also equation (349) which was derived by an approximative method.

† Compare equation (360), obtained by an approximative method.

If V_q is a continuous increasing function of ϵ_q, commencing with $V_q = 0$, the average value in a canonical ensemble of any function of ϵ_q, either alone or with the modulus and the external coördinates, is given by equation (275), which is identical with (357) except that ϵ, ϕ, and ψ have the suffix $(\ )_q$. The equation may be transformed so as to give an equation identical with (359) except for the suffixes. If we add the same suffixes to equation (361), the finite value of its members will determine the possibility of the canonical distribution.

From these data, it is easy to derive equations similar to (360), (362)–(372), except that the conditions of their validity must be differently stated. The equation

$$\overline{e^{-\phi_q} V_q} = \Theta$$

requires only the condition already mentioned with respect to V_q. This equation corresponds to (362), which is subject to no restriction with respect to the value of n. We may observe, however, that V will always satisfy a condition similar to that mentioned with respect to V_q.

If V_q satisfies the condition mentioned, and e^{ϕ_q} a similar condition, *i. e.*, if e^{ϕ_q} is a continuous increasing function of ϵ_q, commencing with the value $\epsilon^{\phi_q} = 0$, equations will hold similar to those given for the case when $n > 2$, viz., similar to (360), (364)–(368). Especially important is

$$\overline{\frac{d\phi_q}{d\epsilon_q}} = \frac{1}{\Theta}.$$

If V_q, e^{ϕ_q} (or $dV_q/d\epsilon_q$), $d^2V_q/d\epsilon_q{}^2$ all satisfy similar conditions, we shall have an equation similar to (369), which was subject to the condition $n > 4$. And if $d^3V_q/d\epsilon_q{}^3$ also satisfies a similar condition, we shall have an equation similar to (372), for which the condition was $n > 6$. Finally, if V_q and h successive differential coefficients satisfy conditions of the kind mentioned, we shall have equations like (370) and (371) for which the condition was $n > 2h$.

These conditions take the place of those given above relating to n. In fact, we might give conditions relating to the differential coefficients of V, similar to those given relating to the differential coefficients of V_q, instead of the conditions relating to n, for the validity of equations (360), (363)–(372). This would somewhat extend the application of the equations.

CHAPTER X.

ON A DISTRIBUTION IN PHASE CALLED MICROCANONICAL IN WHICH ALL THE SYSTEMS HAVE THE SAME ENERGY.

An important case of statistical equilibrium is that in which all systems of the ensemble have the same energy. We may arrive at the notion of a distribution which will satisfy the necessary conditions by the following process. We may suppose that an ensemble is distributed with a uniform density-in-phase between two limiting values of the energy, ϵ' and ϵ'', and with density zero outside of those limits. Such an ensemble is evidently in statistical equilibrium according to the criterion in Chapter IV, since the density-in-phase may be regarded as a function of the energy. By diminishing the difference of ϵ' and ϵ'', we may diminish the differences of energy in the ensemble. The limit of this process gives us a permanent distribution in which the energy is constant.

We should arrive at the same result, if we should make the density any function of the energy between the limits ϵ' and ϵ'', and zero outside of those limits. Thus, the limiting distribution obtained from the part of a canonical ensemble between two limits of energy, when the difference of the limiting energies is indefinitely diminished, is independent of the modulus, being determined entirely by the energy, and is identical with the limiting distribution obtained from a uniform density between limits of energy approaching the same value.

We shall call the limiting distribution at which we arrive by this process *microcanonical.*

We shall find however, in certain cases, that for certain values of the energy, viz., for those for which e^{Φ} is infinite,

this process fails to define a limiting distribution in any such distinct sense as for other values of the energy. The difficulty is not in the process, but in the nature of the case, being entirely analogous to that which we meet when we try to find a canonical distribution in cases when ψ becomes infinite. We have not regarded such cases as affording true examples of the canonical distribution, and we shall not regard the cases in which e^{Φ} is infinite as affording true examples of the microcanonical distribution. We shall in fact find as we go on that in such cases our most important formulae become illusory.

The use of formulae relating to a canonical ensemble which contain $e^{\Phi}\,d\epsilon$ instead of $dp_1 \ldots dq_n$, as in the preceding chapters, amounts to the considëration of the ensemble as divided into an infinity of microcanonical elements.

From a certain point of view, the microcanonical distribution may seem more simple than the canonical, and it has perhaps been more studied, and been regarded as more closely related to the fundamental notions of thermodynamics. To this last point we shall return in a subsequent chapter. It is sufficient here to remark that analytically the canonical distribution is much more manageable than the microcanonical.

We may sometimes avoid difficulties which the microcanonical distribution presents by regarding it as the result of the following process, which involves conceptions less simple but more amenable to analytical treatment. We may suppose an ensemble distributed with a density proportional to

$$e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}},$$

where ω and ϵ' are constants, and then diminish indefinitely the value of the constant ω. Here the density is nowhere zero until we come to the limit, but at the limit it is zero for all energies except ϵ'. We thus avoid the analytical complication of discontinuities in the value of the density, which require the use of integrals with inconvenient limits.

In a microcanonical ensemble of systems the energy (ϵ) is constant, but the kinetic energy (ϵ_p) and the potential energy

(ϵ_q) vary in the different systems, subject of course to the condition

$$\epsilon_p + \epsilon_q = \epsilon = \text{constant.} \tag{373}$$

Our first inquiries will relate to the division of energy into these two parts, and to the average values of functions of ϵ_p and ϵ_q.

We shall use the notation $\overline{u}|_\epsilon$ to denote an average value in a microcanonical ensemble of energy ϵ. An average value in a canonical ensemble of modulus Θ, which has hitherto been denoted by $\overline{u}$, we shall in this chapter denote by $\overline{u}|_\Theta$, to distinguish more clearly the two kinds of averages.

The extension-in-phase within any limits which can be given in terms of ϵ_p and ϵ_q may be expressed in the notations of the preceding chapter by the double integral

$$\int\int dV_p\, dV_q$$

taken within those limits. If an ensemble of systems is distributed within those limits with a uniform density-in-phase, the average value in the ensemble of any function (u) of the kinetic and potential energies will be expressed by the quotient of integrals

$$\frac{\int\int u\, dV_p\, dV_q}{\int\int dV_p\, dV_q}$$

Since $dV_p = e^{\phi_p}\, d\epsilon_p$, and $d\epsilon_p = d\epsilon$ when ϵ_q is constant, the expression may be written

$$\frac{\int\int u\, e^{\phi_p}\, d\epsilon\, dV_q}{\int\int e^{\phi_p}\, d\epsilon\, dV_q}$$

To get the average value of u in an ensemble distributed microcanonically with the energy ϵ, we must make the integrations cover the extension-in-phase between the energies ϵ and $\epsilon + d\epsilon$. This gives

$$\overline{u}\Big]_\epsilon = \frac{d\epsilon \int_{V_q=0}^{\epsilon_q=\epsilon} u\, e^{\phi_p} dV_q}{d\epsilon \int_{V_q=0}^{\epsilon_q=\epsilon} e^{\phi_p} dV_q}$$

But by (299) the value of the integral in the denominator is e^{ϕ}. We have therefore

$$\overline{u}\Big]_\epsilon = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} u\, e^{\phi_p} dV_q, \tag{374}$$

where e^{ϕ_p} and V_q are connected by equation (373), and u, if given as function of ϵ_p, or of ϵ_p and ϵ_q, becomes in virtue of the same equation a function of ϵ_q alone.

We shall assume that e^{ϕ} has a finite value. If $n > 1$, it is evident from equation (305) that e^{ϕ} is an increasing function of ϵ, and therefore cannot be infinite for one value of ϵ without being infinite for all greater values of ϵ, which would make $-\psi$ infinite.* When $n > 1$, therefore, if we assume that e^{ϕ} is finite, we only exclude such cases as we found necessary to exclude in the study of the canonical distribution. But when $n = 1$, cases may occur in which the canonical distribution is perfectly applicable, but in which the formulae for the microcanonical distribution become illusory, for particular values of ϵ, on account of the infinite value of e^{ϕ}. Such failing cases of the microcanonical distribution *for particular values of the energy* will not prevent us from regarding the canonical ensemble as consisting of an infinity of microcanonical ensembles.†

* See equation (322).

† An example of the failing case of the microcanonical distribution is afforded by a material point, under the influence of gravity, and constrained to remain in a vertical circle. The failing case occurs when the energy is just sufficient to carry the material point to the highest point of the circle.

It will be observed that the difficulty is inherent in the nature of the case, and is quite independent of the mathematical formulae. The nature of the difficulty is at once apparent if we try to distribute a finite number of

From the last equation, with (298), we get

$$\overline{e^{-\phi_p} V_p}\Big|_\epsilon = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} V_p \, d V_q = e^{-\phi} V. \tag{375}$$

But by equations (288) and (289)

$$e^{-\phi_p} V_p = \frac{2}{n} \epsilon_p. \tag{376}$$

Therefore

$$e^{-\phi} V = \overline{e^{-\phi_p} V_p}\Big|_\epsilon = \frac{2}{n} \overline{\epsilon_p}\Big|_\epsilon. \tag{377}$$

Again, with the aid of equation (301), we get

$$\overline{\frac{d\phi_p}{d\epsilon_p}}\Big|_\epsilon = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} \frac{d\phi_p}{d\epsilon_p} e^{\phi_p} d V_q = \frac{d\phi}{d\epsilon}, \tag{378}$$

if $n > 2$. Therefore, by (289),

$$\frac{d\phi}{d\epsilon} = \overline{\frac{d\phi_p}{d\epsilon_p}}\Big|_\epsilon = \left(\frac{n}{2} - 1\right) \overline{\epsilon_p^{-1}}\Big|_\epsilon, \quad \text{if} \quad n > 2. \tag{379}$$

These results are interesting on account of the relations of the functions $e^{-\phi} V$ and $\frac{d\phi}{d\epsilon}$ to the notion of temperature in thermodynamics, — a subject to which we shall return hereafter. They are particular cases of a general relation easily deduced from equations (306), (374), (288) and (289). We have

$$\frac{d^h V}{d\epsilon^h} = \int_{V_q=0}^{\epsilon_q=\epsilon} \frac{d^h V_p}{d\epsilon_p^{\,h}} d V_q, \quad \text{if} \quad h < \tfrac{1}{2} n + 1.$$

The equation may be written

$$e^{-\phi} \frac{d^h V}{d\epsilon^h} = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} e^{-\phi_p} \frac{d^h V_p}{d\epsilon_p^{\,h}} e^{\phi_p} d V_q.$$

material points with this particular value of the energy as nearly as possible in statistical equilibrium, or if we ask: What is the probability that a point taken at random from an ensemble in statistical equilibrium with this value of the energy will be found in any specified part of the circle?

We have therefore

$$e^{-\phi}\frac{d^h V}{d\epsilon^h} = \overline{e^{-\phi_p}\frac{d^h V_p}{d\epsilon_p{}^h}}\Big|_\epsilon = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n-h+1)}\overline{\epsilon_p{}^{1-h}}\Big|_\epsilon, \quad (380)$$

if $h < \frac{1}{2}n + 1$. For example, when n is even, we may make $h = \frac{1}{2}n$, which gives, with (307),

$$(2\pi)^{\frac{n}{2}} e^{-\phi}(V_q)_{\epsilon_q=\epsilon} = \Gamma(\tfrac{1}{2}n)\,\overline{\epsilon_p{}^{1-\frac{n}{2}}}\Big|_\epsilon. \quad (381)$$

Since any canonical ensemble of systems may be regarded as composed of microcanonical ensembles, if any quantities u and v have the same average values in every microcanonical ensemble, they will have the same values in every canonical ensemble. To bring equation (380) formally under this rule, we may observe that the first member being a function of ϵ is a constant value in a microcanonical ensemble, and therefore identical with its average value. We get thus the general equation

$$\overline{e^{-\phi}\frac{d^h V}{d\epsilon^h}}\Big|_\Theta = \overline{e^{-\phi_p}\frac{d^h V_p}{d\epsilon_p{}^h}}\Big|_\Theta = \frac{\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n-h+1)}\overline{\epsilon_p{}^{1-h}}\Big|_\Theta = \Theta^{1-h}, \quad (382)$$

if $h < \frac{1}{2}n + 1$.* The equations

$$\Theta = \overline{e^{-\phi}V}\Big|_\Theta = \overline{e^{-\phi_p}V_p}\Big|_\Theta = \frac{2}{n}\overline{\epsilon_p}\Big|_\Theta, \quad (383)$$

$$\frac{1}{\Theta} = \overline{\frac{d\phi}{d\epsilon}}\Big|_\Theta = \overline{\frac{d\phi_p}{d\epsilon_p}}\Big|_\Theta = \left(\frac{n}{2}-1\right)\overline{\epsilon_p{}^{-1}}\Big|_\Theta, \quad (384)$$

may be regarded as particular cases of the general equation. The last equation is subject to the condition that $n > 2$.

The last two equations give for a canonical ensemble, if $n > 2$,

$$\left(1-\frac{2}{n}\right)\overline{\epsilon_p}\Big|_\Theta\,\overline{\epsilon_p{}^{-1}}\Big|_\Theta = 1. \quad (385)$$

The corresponding equations for a microcanonical ensemble give, if $n > 2$,

$$\left(1-\frac{2}{n}\right)\overline{\epsilon_p}\Big|_\epsilon\,\overline{\epsilon_p{}^{-1}}\Big|_\epsilon = \frac{d\phi}{d\log V}, \quad (386)$$

* See equation (292).

which shows that $d\phi \; d \log V$ approaches the value unity when n is very great.

If a system consists of two parts, having separate energies, we may obtain equations similar in form to the preceding, which relate to the system as thus divided.* We shall distinguish quantities relating to the parts by letters with suffixes, the same letters without suffixes relating to the whole system. The extension-in-phase of the whole system within any given limits of the energies may be represented by the double integral

$$\int\int dV_1 dV_2$$

taken within those limits, as appears at once from the definitions of Chapter VIII. In an ensemble distributed with uniform density within those limits, and zero density outside, the average value of any function of ϵ_1 and ϵ_2 is given by the quotient

$$\frac{\int\int u dV_1 dV_2}{\int\int dV_1 dV_2}$$

which may also be written †

$$\frac{\int\int u \, e^{\phi_1} d\epsilon \, dV_2}{\int\int e^{\phi_1} d\epsilon \, dV_2}$$

If we make the limits of integration ϵ and $\epsilon + d\epsilon$, we get the

* If this condition is rigorously fulfilled, the parts will have no influence on each other, and the ensemble formed by distributing the whole microcanonically is too arbitrary a conception to have a real interest. The principal interest of the equations which we shall obtain will be in cases in which the condition is approximately fulfilled. But for the purposes of a theoretical discussion, it is of course convenient to make such a condition absolute. Compare Chapter IV, pp. 35 ff., where a similar condition is considered in connection with canonical ensembles.

† Where the analytical transformations are identical in form with those on the preceding pages, it does not appear necessary to give all the steps with the same detail.

average value of u in an ensemble in which the whole system is microcanonically distributed in phase, viz.,

$$\overline{u}\Big|_{\epsilon} = e^{-\phi} \int_{V_2=0}^{\epsilon_2=\epsilon} u\, e^{\phi_1}\, dV_2, \tag{387}$$

where ϕ_1 and V_2 are connected by the equation

$$\epsilon_1 + \epsilon_2 = \text{constant} = \epsilon, \tag{388}$$

and u, if given as function of ϵ_1, or of ϵ_1 and ϵ_2, becomes in virtue of the same equation a function of ϵ_2 alone.*
Thus

$$\overline{e^{-\phi_1} V_1}\Big|_{\epsilon} = e^{-\phi} \int_{V_2=0}^{\epsilon_2=\epsilon} V_1\, dV_2, \tag{389}$$

$$e^{-\phi} V = \overline{e^{-\phi_1} V_1}\Big|_{\epsilon} = \overline{e^{-\phi_2} V_2}\Big|_{\epsilon}. \tag{390}$$

This requires a similar relation for canonical averages

$$\Theta = \overline{e^{-\phi} V}\Big|_{\Theta} = \overline{e^{-\phi_1} V_1}\Big|_{\Theta} = \overline{e^{-\phi_2} V_2}\Big|_{\Theta}. \tag{391}$$

Again

$$\overline{\frac{d\phi_1}{d\epsilon_1}}\Big|_{\epsilon} = e^{-\phi} \int_{V_2=0}^{\epsilon_2=\epsilon} \frac{d\phi_1}{d\epsilon_1} e^{\phi_1}\, dV_2. \tag{392}$$

But if $n_1 > 2$, e^{ϕ_1} vanishes for $V_1 = 0$,† and

$$\frac{d}{d\epsilon} e^{\phi} = \frac{d}{d\epsilon} \int_{V_2=0}^{\epsilon_2=\epsilon} e^{\phi_1}\, dV_2 = \int_{V_2=0}^{\epsilon_2=\epsilon} \frac{d\phi_1}{d\epsilon_1} e^{\phi_1}\, dV_2. \tag{393}$$

Hence, if $n_1 > 2$, and $n_2 > 2$,

$$\frac{d\phi}{d\epsilon} = \overline{\frac{d\phi_1}{d\epsilon_1}}\Big|_{\epsilon} = \overline{\frac{d\phi_2}{d\epsilon_2}}\Big|_{\epsilon}, \tag{394}$$

* In the applications of the equation (387), we cannot obtain all the results corresponding to those which we have obtained from equation (374), because ϕ_p is a known function of ϵ_p, while ϕ_1 must be treated as an arbitrary function of ϵ_1, or nearly so.

† See Chapter VIII, equations (305) and (316).

and
$$\frac{1}{\Theta} = \overline{\frac{d\phi}{d\epsilon}}\Big|_{\Theta} = \overline{\frac{d\phi_1}{d\epsilon_1}}\Big|_{\Theta} = \overline{\frac{d\phi_2}{d\epsilon_2}}\Big|_{\Theta}. \tag{395}$$

We have compared certain functions of the energy of the whole system with average values of similar functions of the kinetic energy of the whole system, and with average values of similar functions of the whole energy of a part of the system. We may also compare the same functions with average values of the kinetic energy of a part of the system.

We shall express the total, kinetic, and potential energies of the whole system by ϵ, ϵ_p, and ϵ_q, and the kinetic energies of the parts by ϵ_{1p} and ϵ_{2p}. These kinetic energies are necessarily separate: we need not make any supposition concerning potential energies. The extension-in-phase within any limits which can be expressed in terms of ϵ_q, ϵ_{1p}, ϵ_{2p} may be represented in the notations of Chapter VIII by the triple integral

$$\int\int\int dV_{1p}\, dV_{2p}\, dV_q$$

taken within those limits. And if an ensemble of systems is distributed with a uniform density within those limits, the average value of any function of ϵ_q, ϵ_{1p}, ϵ_{2p} will be expressed by the quotient

$$\frac{\int\int\int u\, dV_{1p}\, dV_{2p}\, dV_q}{\int\int\int dV_{1p}\, dV_{2p}\, dV_q}$$

or
$$\frac{\int\int\int u e^{\phi_{1p}} d\epsilon\, dV_{2p}\, dV_q}{\int\int\int e^{\phi_{1p}} d\epsilon\, dV_{2p}\, dV_q}$$

To get the average value of u for a microcanonical distribution, we must make the limits ϵ and $\epsilon + d\epsilon$. The denominator in this case becomes $\epsilon^{\phi}\, d\epsilon$, and we have

$$\overline{u}\Big|_{\epsilon} = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} \int_{\epsilon_{2p}=0}^{\epsilon_{2p}=\epsilon-\epsilon_q} u e^{\phi_{1p}}\, dV_{2p}\, dV_q, \tag{396}$$

where ϕ_{1p}, V_{2p}, and V_q are connected by the equation

$$\epsilon_{1p} + \epsilon_{2p} + \epsilon_q = \text{constant} = \epsilon.$$

Accordingly

$$\overline{e^{-\phi_{1p}} V_{1p}}\Big|_\epsilon = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} \int_{\epsilon_{2p}=0}^{\epsilon_{2p}=\epsilon-\epsilon_q} V_{1p}\, d\, V_{2p}\, d\, V_q = e^{-\phi}\, V, \quad (397)$$

and we may write

$$e^{-\phi} V = \overline{e^{-\phi_{1p}} V_{1p}}\Big|_\epsilon = \overline{e^{-\phi_{2p}} V_{2p}}\Big|_\epsilon = \frac{2}{n_1} \overline{\epsilon_{1p}}\Big|_\epsilon = \frac{2}{n_2} \overline{\epsilon_{2p}}\Big|_\epsilon, \quad (398)$$

and

$$\Theta = \overline{e^{-\phi} V}\Big|_\Theta = \overline{e^{-\phi_{1p}} V_{1p}}\Big|_\Theta = \overline{e^{-\phi_{2p}} V_{2p}}\Big|_\Theta = \frac{2}{n_1} \overline{\epsilon_{1p}}\Big|_\Theta = \frac{2}{n_2} \overline{\epsilon_{2p}}\Big|_\Theta. \quad (399)$$

Again, if $n_1 > 2$,

$$\overline{\frac{d\phi_{1p}}{d\epsilon_{1p}}}\Big| = e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} \int_{\epsilon_{2p}=0}^{\epsilon_{2p}=\epsilon-\epsilon_q} \frac{d\phi_{1p}}{d\epsilon_{1p}} e^{\phi_{1p}}\, d\, V_{2p}\, d\, V_q$$

$$= e^{-\phi} \int_{V_q=0}^{\epsilon_q=\epsilon} \frac{d\epsilon^{\phi_p}}{d\epsilon_p}\, d\, V_q = e^{-\phi} \frac{d\epsilon^{\phi}}{d\epsilon} = \frac{d\phi}{d\epsilon}. \quad (400)$$

Hence, if $n_1 > 2$, and $n_2 > 2$,

$$\frac{d\phi}{d\epsilon} = \overline{\frac{d\phi_{1p}}{d\epsilon_{1p}}}\Big|_\epsilon = \overline{\frac{d\phi_{2p}}{d\epsilon_{2p}}}\Big|_\epsilon = (\tfrac{1}{2} n_1 - 1)\, \overline{\epsilon_{1p}^{-1}}\Big|_\epsilon = (\tfrac{1}{2} n_2 - 1)\, \overline{\epsilon_{2p}^{-1}}\Big|_\epsilon, \quad (401)$$

$$\frac{1}{\Theta} = \overline{\frac{d\phi}{d\epsilon}}\Big|_\Theta = \overline{\frac{d\phi_{1p}}{d\epsilon_{1p}}}\Big|_\Theta = \overline{\frac{d\phi_{2p}}{d\epsilon_{2p}}}\Big|_\Theta = (\tfrac{1}{2} n_1 - 1)\, \overline{\epsilon_{1p}^{-1}}\Big|_\Theta = (\tfrac{1}{2} n_2 - 1)\, \overline{\epsilon_{2p}^{-1}}\Big|_\Theta. \quad (402)$$

We cannot apply the methods employed in the preceding pages to the microcanonical averages of the (generalized) forces A_1, A_2, etc., exerted by a system on external bodies, since these quantities are not functions of the energies, either kinetic or potential, of the whole or any part of the system. We may however use the method described on page 116.

Let us imagine an ensemble of systems distributed in phase according to the index of probability

$$c - \frac{(\epsilon - \epsilon')^2}{\omega^2},$$

where ϵ' is any constant which is a possible value of the energy, except only the least value which is consistent with the values of the external coördinates, and c and ω are other constants. We have therefore

$$\int_{\text{phases}}^{\text{all}} \cdots \int e^{c - \frac{(\epsilon-\epsilon')^2}{\omega^2}} dp_1 \ldots dq_n = 1, \tag{403}$$

or
$$e^{-c} = \int_{\text{phases}}^{\text{all}} \cdots \int e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}} dp_1 \ldots dq_n, \tag{404}$$

or again
$$e^{-c} = \int_{V=0}^{\epsilon=\infty} e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi} d\epsilon. \tag{405}$$

From (404) we have

$$\frac{de^{-c}}{da_1} = \int_{\text{phases}}^{\text{all}} \cdots \int 2\frac{\epsilon - \epsilon'}{\omega^2} A_1 e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}} dp_1 \ldots dq_n$$

$$= \int_{V=0}^{\epsilon=\infty} 2\frac{\epsilon - \epsilon'}{\omega^2} \overline{A_1}|_\epsilon \, e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi} d\epsilon, \tag{406}$$

where $\overline{A_1}|_\epsilon$ denotes the average value of A_1 in those systems of the ensemble which have any same energy ϵ. (This is the same thing as the average value of A_1 in a microcanonical ensemble of energy ϵ.) The validity of the transformation is evident, if we consider separately the part of each integral which lies between two infinitesimally differing limits of energy. Integrating by parts, we get

$$\frac{de^{-c}}{da_1} = -\left[\overline{A_1}\Big|_\epsilon e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi}\right]_{V=0}^{\epsilon=\infty} + \int_{V=0}^{\epsilon=\infty}\left(\frac{d\overline{A_1}\Big|_\epsilon}{d\epsilon} + \overline{A_1}\Big|_\epsilon \frac{d\phi}{d\epsilon}\right) e^{-\frac{(\epsilon-\epsilon')}{\omega^2}+\phi}\, d\epsilon. \qquad (407)$$

Differentiating (405), we get

$$\frac{de^{-c}}{da_1} = \int_{V=0}^{\epsilon=\infty} \frac{d\phi}{da_1} e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi}\, d\epsilon - \left(e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi}\frac{d\epsilon_a}{da_1}\right)_{V=0} \qquad (408)$$

where ϵ_a denotes the least value of ϵ consistent with the external coördinates. The last term in this equation represents the part of de^{-c}/da_1 which is due to the variation of the lower limit of the integral. It is evident that the expression in the brackets will vanish at the upper limit. At the lower limit, at which $\epsilon_p = 0$, and ϵ_q has the least value consistent with the external coördinates, the average sign on $\overline{A_1}\Big|_\epsilon$ is superfluous, as there is but one value of A_1 which is represented by $-d\epsilon_a/da_1$. Exceptions may indeed occur for particular values of the external coördinates, at which $d\epsilon_a/da_1$ receive a finite increment, and the formula becomes illusory. Such particular values we may for the moment leave out of account. The last term of (408) is therefore equal to the first term of the second member of (407). (We may observe that both vanish when $n > 2$ on account of the factor e^{ϕ}.)

We have therefore from these equations

$$\int_{V=0}^{\epsilon=\infty}\left(\frac{d\overline{A_1}\Big|_\epsilon}{d\epsilon} + \overline{A_1}\Big|_\epsilon \frac{d\phi}{d\epsilon}\right) e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi}\, d\epsilon = \int_{V=0}^{\epsilon=\infty}\frac{d\phi}{da_1} e^{-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi}\, d\epsilon,$$

or

$$\int_{V=0}^{\epsilon=\infty}\left(\frac{d\overline{A_1}\Big|_\epsilon}{d\epsilon} + \overline{A_1}\Big|_\epsilon \frac{d\phi}{d\epsilon} - \frac{d\phi}{da_1}\right) e^{c-\frac{(\epsilon-\epsilon')^2}{\omega^2}+\phi}\, d\epsilon = 0. \qquad (409)$$

That is: the average value in the ensemble of the quantity represented by the principal parenthesis is zero. This must

be true for any value of ω. If we diminish ω, the average value of the parenthesis at the limit when ω vanishes becomes identical with the value for $\epsilon = \epsilon'$. But this may be any value of the energy, except the least possible. We have therefore

$$\frac{d\overline{A_1}|_\epsilon}{d\epsilon} + \overline{A_1}|_\epsilon \frac{d\phi}{d\epsilon} - \frac{d\phi}{da_1} = 0, \tag{410}$$

unless it be for the least value of the energy consistent with the external coördinates, or for particular values of the external coördinates. But the value of any term of this equation as determined for particular values of the energy and of the external coördinates is not distinguishable from its value as determined for values of the energy and external coördinates indefinitely near those particular values. The equation therefore holds without limitation. Multiplying by e^ϕ, we get

$$e^\phi \frac{d\overline{A_1}|_\epsilon}{d\epsilon} + \overline{A_1}|_\epsilon\, e^\phi \frac{d\phi}{d\epsilon} = e^\phi \frac{d\phi}{da_1} = \frac{de^\phi}{da_1} = \frac{d^2V}{da_1\, d\epsilon}. \tag{411}$$

The integral of this equation is

$$\overline{A_1}|_\epsilon e^\phi = \frac{dV}{da_1} + F_1, \tag{412}$$

where F_1 is a function of the external coördinates. We have an equation of this form for each of the external coördinates. This gives, with (266), for the complete value of the differential of V

$$dV = e^\phi\, d\epsilon + (e^\phi \overline{A_1}|_\epsilon - F_1)\, da_1 + (e^\phi \overline{A_2}|_\epsilon - F_2)\, da_2 + \text{etc.}, \tag{413}$$

or

$$dV = e^\phi\, (d\epsilon + \overline{A_1}|_\epsilon\, da_1 + \overline{A_2}|_\epsilon\, da_2 + \text{etc.}) - F_1\, da_1 - F_2\, da_2 - \text{etc.} \tag{414}$$

To determine the values of the functions F_1, F_2, etc., let us suppose a_1, a_2, etc. to vary arbitrarily, while ϵ varies so as always to have the least value consistent with the values of the external coördinates. This will make $V = 0$, and $dV = 0$. If $n < 2$, we shall have also $e^\phi = 0$, which will give

$$F_1 = 0, \quad F_2 = 0, \quad \text{etc.} \tag{415}$$

The result is the same for any value of n. For in the variations considered the kinetic energy will be constantly zero, and the potential energy will have the least value consistent with the external coördinates. The condition of the least possible potential energy may limit the ensemble at each instant to a single configuration, or it may not do so; but in any case the values of A_1, A_2, etc. will be the same at each instant for all the systems of the ensemble,* and the equation

$$d\epsilon + A_1 da_1 + A_2 da_2 + \text{etc.} = 0$$

will hold for the variations considered. Hence the functions F_1, F_2, etc. vanish in any case, and we have the equation

$$dV = e^{\phi} d\epsilon + e^{\phi} \overline{A_1}|_{\epsilon} da_1 + e^{\phi} \overline{A_2}|_{\epsilon} da_2 + \text{etc.}, \tag{416}$$

or

$$d \log V = \frac{d\epsilon + \overline{A_1}|_{\epsilon} da_1 + \overline{A_2}|_{\epsilon} da_2 + \text{etc.}}{e^{-\phi} V}, \tag{417}$$

or again

$$d\epsilon = e^{-\phi} V d \log V - \overline{A_1}|_{\epsilon} da_1 - \overline{A_2}|_{\epsilon} da_2 - \text{etc.} \tag{418}$$

It will be observed that the two last equations have the form of the fundamental differential equations of thermodynamics, $e^{-\phi} V$ corresponding to temperature and $\log V$ to entropy. We have already observed properties of $e^{-\phi} V$ suggestive of an analogy with temperature.† The significance of these facts will be discussed in another chapter.

The two last equations might be written more simply

$$dV = \frac{d\epsilon + \overline{A_1}|_{\epsilon} da_1 + \overline{A_2}|_{\epsilon} da_2 + \text{etc.}}{e^{-\phi}},$$

$$d\epsilon = e^{-\phi} dV - \overline{A_1}|_{\epsilon} da_1 - \overline{A_2}|_{\epsilon} da_2 - \text{etc.},$$

and still have the form analogous to the thermodynamic equations, but $e^{-\phi}$ has nothing like the analogies with temperature which we have observed in $e^{-\phi} V$.

* This statement, as mentioned before, may have exceptions for particular values of the external coördinates. This will not invalidate the reasoning, which has to do with varying values of the external coördinates.

† See Chapter IX, page 111; also this chapter, page 119.

CHAPTER XI.

MAXIMUM AND MINIMUM PROPERTIES OF VARIOUS DISTRIBUTIONS IN PHASE.

In the following theorems we suppose, as always, that the systems forming an ensemble are identical in nature and in the values of the external coördinates, which are here regarded as constants.

Theorem I. If an ensemble of systems is so distributed in phase that the index of probability is a function of the energy, the average value of the index is less than for any other distribution in which the distribution in energy is unaltered.

Let us write η for the index which is a function of the energy, and $\eta + \Delta\eta$ for any other which gives the same distribution in energy. It is to be proved that

$$\int_{\text{phases}}^{\text{all}} \cdots \int (\eta + \Delta\eta)\, e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n > \int_{\text{phases}}^{\text{all}} \cdots \int \eta\, e^{\eta}\, dp_1 \ldots dq_n, \quad (419)$$

where η is a function of the energy, and $\Delta\eta$ a function of the phase, which are subject to the conditions that

$$\int_{\text{phases}}^{\text{all}} \cdots \int e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n = \int_{\text{phases}}^{\text{all}} \cdots \int e^{\eta}\, dp_1 \ldots dq_n = 1, \quad (420)$$

and that for any value of the energy (ϵ')

$$\int_{\epsilon=\epsilon'}^{\epsilon=\epsilon'+d\epsilon'} \cdots \int e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n = \int_{\epsilon=\epsilon'}^{\epsilon=\epsilon'+d\epsilon'} \cdots \int e^{\eta} dp_1 \ldots dq_n. \quad (421)$$

Equation (420) expresses the general relations which η and $\eta + \Delta\eta$ must satisfy in order to be indices of any distributions, and (421) expresses the condition that they give the same distribution in energy.

Since η is a function of the energy, and may therefore be regarded as a constant within the limits of integration of (421), we may multiply by η under the integral sign in both members, which gives

$$\int_{\epsilon=\epsilon'}^{\epsilon=\epsilon'+d\epsilon'} \cdots \int \eta e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n = \int_{\epsilon=\epsilon'}^{\epsilon=\epsilon'+d\epsilon'} \cdots \int \eta e^{\eta}\, dp_1 \ldots dq_n.$$

Since this is true within the limits indicated, and for every value of ϵ', it will be true if the integrals are taken for all phases. We may therefore cancel the corresponding parts of (419), which gives

$$\int_{\text{phases}}^{\text{all}} \cdots \int \Delta\eta\, e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n > 0. \tag{422}$$

But by (420) this is equivalent to

$$\int_{\text{phases}}^{\text{all}} \cdots \int (\Delta\eta\, e^{\Delta\eta} + 1 - e^{\Delta\eta}) e^{\eta}\, dp_1 \ldots dq_n > 0. \tag{423}$$

Now $\Delta\eta\, e^{\Delta\eta} + 1 - e^{\Delta\eta}$ is a decreasing function of $\Delta\eta$ for negative values of $\Delta\eta$, and an increasing function of $\Delta\eta$ for positive values of $\Delta\eta$. It vanishes for $\Delta\eta = 0$. The expression is therefore incapable of a negative value, and can have the value 0 only for $\Delta\eta = 0$. The inequality (423) will hold therefore unless $\Delta\eta = 0$ for all phases. The theorem is therefore proved.

Theorem II. If an ensemble of systems is canonically distributed in phase, the average index of probability is less than in any other distribution of the ensemble having the same average energy.

For the canonical distribution let the index be $(\psi - \epsilon)/\Theta$, and for another having the same average energy let the index be $(\psi - \epsilon)/\Theta + \Delta\eta$, where $\Delta\eta$ is an arbitrary function of the phase subject only to the limitation involved in the notion of the index, that

$$\int_{\text{phases}}^{\text{all}} \cdots \int e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta}\, dp_1 \ldots dq_n = \int_{\text{phases}}^{\text{all}} \cdots \int e^{\frac{\psi-\epsilon}{\Theta}}\, dp_1 \ldots dq_n = 1, \tag{424}$$

and to that relating to the constant average energy, that

$$\int_{\text{phases}}^{\text{all}} \cdots \int \epsilon\, e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta}\, dp_1 \ldots dq_n = \int_{\text{phases}}^{\text{all}} \cdots \int \epsilon\, e^{\frac{\psi-\epsilon}{\Theta}}\, dp_1 \ldots dq_n. \tag{425}$$

It is to be proved that

$$\int_{\text{phases}}^{\text{all}} \cdots \int \left(\frac{\psi}{\Theta} - \frac{\epsilon}{\Theta} + \Delta\eta\right) e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta}\, dp_1 \ldots dq_n >$$

$$\int_{\text{phases}}^{\text{all}} \cdots \int \left(\frac{\psi}{\Theta} - \frac{\epsilon}{\Theta}\right) e^{\frac{\psi-\epsilon}{\Theta}}\, dp_1 \ldots dq_n. \tag{426}$$

Now in virtue of the first condition (424) we may cancel the constant term ψ/Θ in the parentheses in (426), and in virtue of the second condition (425) we may cancel the term ϵ/Θ. The proposition to be proved is thus reduced to

$$\int_{\text{phases}}^{\text{all}} \cdots \int \Delta\eta\, e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta}\, dp_1 \ldots dq_n > 0,$$

which may be written, in virtue of the condition (424),

$$\int_{\text{phases}}^{\text{all}} \cdots \int (\Delta\eta\, e^{\Delta\eta} + 1 - e^{\Delta\eta})\, e^{\frac{\psi-\epsilon}{\Theta}}\, dp_1 \ldots dq_n > 0. \tag{427}$$

In this form its truth is evident for the same reasons which applied to (423).

Theorem III. If Θ is any positive constant, the average value in an ensemble of the expression $\eta + \epsilon/\Theta$ (η denoting as usual the index of probability and ϵ the energy) is less when the ensemble is distributed canonically with modulus Θ, than for any other distribution whatever.

In accordance with our usual notation let us write $(\psi - \epsilon)/\Theta$ for the index of the canonical distribution. In any other distribution let the index be $(\psi - \epsilon)/\Theta + \Delta\eta$.

In the canonical ensemble $\eta + \epsilon/\Theta$ has the constant value ψ/Θ; in the other ensemble it has the value $\psi/\Theta + \Delta\eta$. The proposition to be proved may therefore be written

$$\frac{\psi}{\Theta} < \int_{\text{phases}}^{\text{all}} \cdots \int \left(\frac{\psi}{\Theta} + \Delta\eta\right) e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta} dp_1 \ldots dq_n, \qquad (428)$$

where

$$\int_{\text{phases}}^{\text{all}} \cdots \int e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta} dp_1 \ldots dq_n = \int_{\text{phases}}^{\text{all}} \cdots \int e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \ldots dq_n = 1. \qquad (429)$$

In virtue of this condition, since ψ/Θ is constant, the proposition to be proved reduces to

$$0 < \int_{\text{phases}}^{\text{all}} \cdots \int \Delta\eta\, e^{\frac{\psi-\epsilon}{\Theta}+\Delta\eta} dq_1 \ldots dp_n, \qquad (430)$$

where the demonstration may be concluded as in the last theorem.

If we should substitute for the energy in the preceding theorems any other function of the phase, the theorems, *mutatis mutandis*, would still hold. On account of the unique importance of the energy as a function of the phase, the theorems as given are especially worthy of notice. When the case is such that other functions of the phase have important properties relating to statistical equilibrium, as described in Chapter IV,* the three following theorems, which are generalizations of the preceding, may be useful. It will be sufficient to give them without demonstration, as the principles involved are in no respect different.

Theorem IV. If an ensemble of systems is so distributed in phase that the index of probability is any function of F_1, F_2, etc., (these letters denoting functions of the phase,) the average value of the index is less than for any other distribution in phase in which the distribution with respect to the functions F_1, F_2, etc. is unchanged.

* See pages 37–41.

Theorem V. If an ensemble of systems is so distributed in phase that the index of probability is a linear function of F_1, F_2, etc., (these letters denoting functions of the phase,) the average value of the index is less than for any other distribution in which the functions F_1, F_2, etc. have the same average values.

Theorem VI. The average value in an ensemble of systems of $\eta + F$ (where η denotes as usual the index of probability and F any function of the phase) is less when the ensemble is so distributed that $\eta + F$ is constant than for any other distribution whatever.

Theorem VII. If a system which in its different phases constitutes an ensemble consists of two parts, and we consider the average index of probability for the whole system, and also the average indices for each of the parts taken separately, the sum of the average indices for the parts will be either less than the average index for the whole system, or equal to it, but cannot be greater. The limiting case of equality occurs when the distribution in phase of each part is independent of that of the other, and only in this case.

Let the coördinates and momenta of the whole system be $q_1 \ldots q_n, p_1, \ldots p_n$, of which $q_1 \ldots q_m p_1, \ldots p_m$ relate to one part of the system, and $q_{m+1}, \ldots q_n, p_{m+1}, \ldots p_n$ to the other. If the index of probability for the whole system is denoted by η, the probability that the phase of an unspecified system lies within any given limits is expressed by the integral

$$\int \ldots \int e^{\eta} dp_1 \ldots dq_n \tag{431}$$

taken for those limits. If we set

$$\int \ldots \int e^{\eta} dp_{m+1} \ldots dp_n dq_{m+1} \ldots dq_n = e^{\eta_1}, \tag{432}$$

where the integrations cover all phases of the second system, and

$$\int \ldots \int e^{\eta} dp_1 \ldots dp_m dq_1 \ldots dq_m = e^{\eta_2}, \tag{433}$$

where the integrations cover all phases of the first system, the integral (431) will reduce to the form

$$\int \dots \int e^{\eta_1} \, dp_1 \dots dp_m dq_1 \dots dq_m, \tag{434}$$

when the limits can be expressed in terms of the coördinates and momenta of the first part of the system. The same integral will reduce to

$$\int \dots \int e^{\eta_2} \, dp_{m+1} \dots dp_n \, dq_{m+1} \dots dq_n, \tag{435}$$

when the limits can be expressed in terms of the coördinates and momenta of the second part of the system. It is evident that η_1 and η_2 are the indices of probability for the two parts of the system taken separately.

The main proposition to be proved may be written

$$\int \dots \int \eta_1 e^{\eta_1} \, dp_1 \dots dq_m + \int \dots \int \eta_2 e^{\eta_2} \, dp_{m+1} \dots dq_n \leqq \int \dots \int \eta \, e^{\eta} \, dp_1 \dots dq_n, \tag{436}$$

where the first integral is to be taken over all phases of the first part of the system, the second integral over all phases of the second part of the system, and the last integral over all phases of the whole system. Now we have

$$\int \dots \int e^{\eta} \, dp_1 \dots dq_n = 1, \tag{437}$$

$$\int \dots \int e^{\eta_1} dp_1 \dots dq_m = 1, \tag{438}$$

and

$$\int \dots \int e^{\eta_2} dp_{m+1} \dots dq_n = 1, \tag{439}$$

where the limits cover in each case all the phases to which the variables relate. The two last equations, which are in themselves evident, may be derived by partial integration from the first.

It appears from the definitions of η_1 and η_2 that (436) may also be written

$$\int \cdots \int \eta_1 e^{\eta} dp_1 \cdots dq_n + \int \cdots \int \eta_2 e^{\eta} dp_1 \cdots dq_n \leqq \int \cdots \int \eta e^{\eta} dp_1 \cdots dq_n, \quad (440)$$

or

$$\int \cdots \int (\eta - \eta_1 - \eta_2) e^{\eta} dp_1 \cdots dq_n \geqq 0,$$

where the integrations cover all phases. Adding the equation

$$\int \cdots \int e^{\eta_1 + \eta_2} dp_1 \cdots dq_n = 1, \quad (441)$$

which we get by multiplying (438) and (439), and subtracting (437), we have for the proposition to be proved

$$\int_{\text{phases}}^{\text{all}} \cdots \int [(\eta - \eta_1 - \eta_2) e^{\eta} + e^{\eta_1 + \eta_2} - e^{\eta}] dp_1 \cdots dq_n \geqq 0. \quad (442)$$

Let

$$u = \eta - \eta_1 - \eta_2. \quad (443)$$

The main proposition to be proved may be written

$$\int_{\text{phases}}^{\text{all}} \cdots \int (u e^{u} + 1 - e^{u}) e^{\eta_1 + \eta_2} dp_1 \cdots dq_n \geqq 0. \quad (444)$$

This is evidently true since the quantity in the parenthesis is incapable of a negative value.* Moreover the sign = can hold only when the quantity in the parenthesis vanishes for all phases, *i. e.*, when $u = 0$ for all phases. This makes $\eta = \eta_1 + \eta_2$ for all phases, which is the analytical condition which expresses that the distributions in phase of the two parts of the system are independent.

Theorem VIII. If two or more ensembles of systems which are identical in nature, but may be distributed differently in phase, are united to form a single ensemble, so that the probability-coefficient of the resulting ensemble is a linear function

* See Theorem I, where this is proved of a similar expression.

of the probability-coefficients of the original ensembles, the average index of probability of the resulting ensemble cannot be greater than the same linear function of the average indices of the original ensembles. It can be equal to it only when the original ensembles are similarly distributed in phase.

Let P_1, P_2, etc. be the probability-coefficients of the original ensembles, and P that of the ensemble formed by combining them; and let N_1, N_2, etc. be the numbers of systems in the original ensembles. It is evident that we shall have

$$P = c_1 P_1 + c_2 P_2 + \text{etc.} = \Sigma(c_1 P_1), \tag{445}$$

where
$$c_1 = \frac{N_1}{\Sigma N_1}, \quad c_2 = \frac{N_2}{\Sigma N_1}, \quad \text{etc.} \tag{446}$$

The main proposition to be proved is that

$$\int_{\text{phases}}^{\text{all}} \cdots \int P \log P \, dp_1 \ldots dq_n \leqq \Sigma \left[c_1 \int_{\text{phases}}^{\text{all}} \cdots \int P_1 \log P_1 \, dp_1 \ldots dq_n \right] \tag{447}$$

or
$$\int_{\text{phases}}^{\text{all}} \cdots \int [\Sigma (c_1 P_1 \log P_1) - P \log P] \, dp_1 \ldots dq_n \geqq 0. \tag{448}$$

If we set

$$Q_1 = P_1 \log P_1 - P_1 \log P - P_1 + P$$

Q_1 will be positive, except when it vanishes for $P_1 = P$. To prove this, we may regard P_1 and P as any positive quantities. Then

$$\left(\frac{dQ_1}{dP_1}\right)_P = \log P_1 - \log P,$$

$$\left(\frac{d^2 Q_1}{dP_1^2}\right)_P = \frac{1}{P_1}.$$

Since Q_1 and dQ_1/dP_1 vanish for $P_1 = P$, and the second differential coefficient is always positive, Q_1 must be positive except when $P_1 = P$. Therefore, if Q_2, etc. have similar definitions,

$$\Sigma (c_1 Q_1) \geqq 0. \tag{449}$$

But since $$\Sigma (c_1 P_1) = P$$

and $$\Sigma c_1 = 1,$$

$$\Sigma (c_1 Q_1) = \Sigma (c_1 P_1 \log P_1) - P \log P. \qquad (450)$$

This proves (448), and shows that the sign = will hold only when

$$P_1 = P, \quad P_2 = P, \quad \text{etc.}$$

for all phases, *i. e.*, only when the distribution in phase of the original ensembles are all identical.

Theorem IX. A uniform distribution of a given number of systems within given limits of phase gives a less average index of probability of phase than any other distribution.

Let η be the constant index of the uniform distribution, and $\eta + \Delta\eta$ the index of some other distribution. Since the number of systems within the given limits is the same in the two distributions we have

$$\int \ldots \int e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n = \int \ldots \int e^{\eta}\, dp_1 \ldots dq_n, \qquad (451)$$

where the integrations, like those which follow, are to be taken within the given limits. The proposition to be proved may be written

$$\int \ldots \int (\eta + \Delta\eta)\, e^{\eta+\Delta\eta}\, dp_1 \ldots dq_n > \int \ldots \int \eta\, e^{\eta}\, dp_1 \ldots dq_n, \qquad (452)$$

or, since η is constant,

$$\int \ldots \int (\eta + \Delta\eta)\, e^{\Delta\eta}\, dp_1 \ldots dq_n > \int \ldots \int \eta\, dp_1 \ldots dq_n. \qquad (453)$$

In (451) also we may cancel the constant factor e^{η}, and multiply by the constant factor $(\eta + 1)$. This gives

$$\int \ldots \int (\eta + 1)\, e^{\Delta\eta}\, dp_1 \ldots dq_n = \int \ldots \int (\eta + 1)\, dp_1 \ldots dq_n.$$

The subtraction of this equation will not alter the inequality to be proved, which may therefore be written

$$\int \ldots \int (\Delta\eta - 1)\, e^{\Delta\eta}\, dp_1 \ldots dq_n > \int \ldots \int -\, dp_1 \ldots dq_n$$

or $$\int \cdots \int (\Delta\eta\, e^{\Delta\eta} - e^{\Delta\eta} + 1)\, dp_1 \cdots dq_n > 0. \qquad (454)$$

Since the parenthesis in this expression represents a positive value, except when it vanishes for $\Delta\eta = 0$, the integral will be positive unless $\Delta\eta$ vanishes everywhere within the limits, which would make the difference of the two distributions vanish. The theorem is therefore proved.

CHAPTER XII.

ON THE MOTION OF SYSTEMS AND ENSEMBLES OF SYSTEMS THROUGH LONG PERIODS OF TIME.

An important question which suggests itself in regard to any case of dynamical motion is whether the system considered will return in the course of time to its initial phase, or, if it will not return exactly to that phase, whether it will do so to any required degree of approximation in the course of a sufficiently long time. To be able to give even a partial answer to such questions, we must know something in regard to the dynamical nature of the system. In the following theorem, the only assumption in this respect is such as we have found necessary for the existence of the canonical distribution.

If we imagine an ensemble of identical systems to be distributed with a uniform density throughout any finite extension-in-phase, the number of the systems which leave the extension-in-phase and will not return to it in the course of time is less than any assignable fraction of the whole number; *provided*, that the total extension-in-phase for the systems considered between two limiting values of the energy is finite, these limiting values being less and greater respectively than any of the energies of the first-mentioned extension-in-phase.

To prove this, we observe that at the moment which we call initial the systems occupy the given extension-in-phase. It is evident that some systems must leave the extension immediately, unless all remain in it forever. Those systems which leave the extension at the first instant, we shall call the *front* of the ensemble. It will be convenient to speak of this front as *generating* the extension-in-phase through which it passes in the course of time, as in geometry a surface is said to

generate the volume through which it passes. In equal times the front generates equal extensions in phase. This is an immediate consequence of the principle of *conservation of extension-in-phase*, unless indeed we prefer to consider it as a slight variation in the expression of that principle. For in two equal short intervals of time let the extensions generated be A and B. (We make the intervals short simply to avoid the complications in the enunciation or interpretation of the principle which would arise when the same extension-in-phase is generated more than once in the interval considered.) Now if we imagine that at a given instant systems are distributed throughout the extension A, it is evident that the same systems will after a certain time occupy the extension B, which is therefore equal to A in virtue of the principle cited. The front of the ensemble, therefore, goes on generating equal extensions in equal times. But these extensions are included in a finite extension, viz., that bounded by certain limiting values of the energy. Sooner or later, therefore, the front must generate phases which it has before generated. Such second generation of the same phases must commence with the initial phases. Therefore a portion at least of the front must return to the original extension-in-phase. The same is of course true of the portion of the ensemble which follows that portion of the front through the same phases at a later time.

It remains to consider how large the portion of the ensemble is, which will return to the original extension-in-phase. There can be no portion of the given extension-in-phase, the systems of which leave the extension and do not return. For we can prove for any portion of the extension as for the whole, that at least a portion of the systems leaving it will return.

We may divide the given extension-in-phase into parts as follows. There may be parts such that the systems within them will never pass out of them. These parts may indeed constitute the whole of the given extension. But if the given extension is very small, these parts will in general be non-existent. There may be parts such that systems within them

will all pass out of the given extension and all return within it. The whole of the given extension-in-phase is made up of parts of these two kinds. This does not exclude the possibility of phases on the boundaries of such parts, such that systems starting with those phases would leave the extension and never return. But in the supposed distribution of an ensemble of systems with a uniform density-in-phase, such systems would not constitute any assignable fraction of the whole number.

These distinctions may be illustrated by a very simple example. If we consider the motion of a rigid body of which one point is fixed, and which is subject to no forces, we find three cases. (1) The motion is periodic. (2) The system will never return to its original phase, but will return infinitely near to it. (3) The system will never return either exactly or approximately to its original phase. But if we consider any extension-in-phase, however small, a system leaving that extension will return to it except in the case called by Poinsot 'singular,' viz., when the motion is a rotation about an axis lying in one of two planes having a fixed position relative to the rigid body. But all such phases do not constitute any true *extension-in-phase* in the sense in which we have defined and used the term.*

In the same way it may be proved that the systems in a canonical ensemble which at a given instant are contained within any finite extension-in-phase will in general return to

* An ensemble of systems distributed in phase is a less simple and elementary conception than a single system. But by the consideration of suitable ensembles instead of single systems, we may get rid of the inconvenience of having to consider exceptions formed by particular cases of the integral equations of motion, these cases simply disappearing when the ensemble is substituted for the single system as a subject of study. This is especially true when the ensemble is distributed, as in the case called canonical, throughout an extension-in-phase. In a less degree it is true of the microcanonical ensemble, which does not occupy any extension-in-phase, (in the sense in which we have used the term,) although it is convenient to regard it as a limiting case with respect to ensembles which do, as we thus gain for the subject some part of the analytical simplicity which belongs to the theory of ensembles which occupy true extensions-in-phase.

that extension-in-phase, if they leave it, the exceptions, *i. e.*, the number which pass out of the extension-in-phase and do not return to it, being less than any assignable fraction of the whole number. In other words, the probability that a system taken at random from the part of a canonical ensemble which is contained within any given extension-in-phase, will pass out of that extension and not return to it, is zero.

A similar theorem may be enunciated with respect to a microcanonical ensemble. Let us consider the fractional part of such an ensemble which lies within any given limits of phase. This fraction we shall denote by F. It is evidently constant in time since the ensemble is in statistical equilibrium. The systems within the limits will not in general remain the same, but some will pass out in each unit of time while an equal number come in. Some may pass out never to return within the limits. But the number which in any time however long pass out of the limits never to return will not bear any finite ratio to the number within the limits at a given instant. For, if it were otherwise, let f denote the fraction representing such ratio for the time T. Then, in the time T, the number which pass out never to return will bear the ratio fF to the whole number in the ensemble, and in a time exceeding $T/(fF)$ the number which pass out of the limits never to return would exceed the total number of systems in the ensemble. The proposition is therefore proved.

This proof will apply to the cases before considered, and may be regarded as more simple than that which was given. It may also be applied to any true case of statistical equilibrium. By a true case of statistical equilibrium is meant such as may be described by giving the general value of the probability that an unspecified system of the ensemble is contained within any given limits of phase.*

* An ensemble in which the systems are material points constrained to move in vertical circles, with just enough energy to carry them to the highest points, cannot afford a true example of statistical equilibrium. For any other value of the energy than the critical value mentioned, we might

Let us next consider whether an ensemble of isolated systems has any tendency in the course of time toward a state of statistical equilibrium.

There are certain functions of phase which are constant in time. The distribution of the ensemble with respect to the values of these functions is necessarily invariable, that is, the number of systems within any limits which can be specified in terms of these functions cannot vary in the course of time. The distribution in phase which without violating this condition gives the least value of the average index of probability of phase ($\bar{\eta}$) is unique, and is that in which the

in various ways describe an ensemble in statistical equilibrium, while the same language applied to the critical value of the energy would fail to do so. Thus, if we should say that the ensemble is so distributed that the probability that a system is in any given part of the circle is proportioned to the time which a single system spends in that part, motion in either direction being equally probable, we should perfectly define a distribution in statistical equilibrium for any value of the energy except the critical value mentioned above, but for this value of the energy all the probabilities in question would vanish unless the highest point is included in the part of the circle considered, in which case the probability is unity, or forms one of its limits, in which case the probability is indeterminate. Compare the foot-note on page 118.

A still more simple example is afforded by the uniform motion of a material point in a straight line. Here the impossibility of statistical equilibrium is not limited to any particular energy, and the canonical distribution as well as the microcanonical is impossible.

These examples are mentioned here in order to show the necessity of caution in the application of the above principle, with respect to the question whether we have to do with a true case of statistical equilibrium.

Another point in respect to which caution must be exercised is that the *part* of an ensemble of which the theorem of the return of systems is asserted should be entirely defined by *limits* within which it is contained, and not by any such condition as that a certain function of phase shall have a given value. This is necessary in order that the part of the ensemble which is considered should be any assignable fraction of the whole. Thus, if we have a canonical ensemble consisting of material points in vertical circles, the theorem of the return of systems may be applied to a part of the ensemble defined as contained in a given part of the circle. But it may not be applied in all cases to a part of the ensemble defined as contained in a given part of the circle and having a given energy. It would, in fact, express the exact opposite of the truth when the given energy is the critical value mentioned above.

index of probability (η) is a function of the functions mentioned.* It is therefore a permanent distribution,† and the only permanent distribution consistent with the invariability of the distribution with respect to the functions of phase which are constant in time.

It would seem, therefore, that we might find a sort of measure of the deviation of an ensemble from statistical equilibrium in the excess of the average index above the minimum which is consistent with the condition of the invariability of the distribution with respect to the constant functions of phase. But we have seen that the index of probability is constant in time for each system of the ensemble. The average index is therefore constant, and we find by this method no approach toward statistical equilibrium in the course of time.

Yet we must here exercise great caution. One function may approach indefinitely near to another function, while some quantity determined by the first does not approach the corresponding quantity determined by the second. A line joining two points may approach indefinitely near to the straight line joining them, while its length remains constant. We may find a closer analogy with the case under consideration in the effect of stirring an incompressible liquid.‡ In space of $2n$ dimensions the case might be made analytically identical with that of an ensemble of systems of n degrees of freedom, but the analogy is perfect in ordinary space. Let us suppose the liquid to contain a certain amount of coloring matter which does not affect its hydrodynamic properties. Now the state in which the density of the coloring matter is uniform, *i. e.*, the state of perfect mixture, which is a sort of state of equilibrium in this respect that the distribution of the coloring matter in space is not affected by the internal motions of the liquid, is characterized by a minimum

* See Chapter XI, Theorem IV.

† See Chapter IV, *sub init.*

‡ By *liquid* is here meant the continuous body of theoretical hydrodynamics, and not anything of the molecular structure and molecular motions of real liquids.

value of the average square of the density of the coloring matter. Let us suppose, however, that the coloring matter is distributed with a variable density. If we give the liquid any motion whatever, subject only to the hydrodynamic law of incompressibility, — it may be a steady flux, or it may vary with the time, — the density of the coloring matter at any same point of the liquid will be unchanged, and the average square of this density will therefore be unchanged. Yet no fact is more familiar to us than that stirring tends to bring a liquid to a state of uniform mixture, or uniform densities of its components, which is characterized by minimum values of the average squares of these densities. It is quite true that in the physical experiment the result is hastened by the process of diffusion, but the result is evidently not dependent on that process.

The contradiction is to be traced to the notion of the *density* of the coloring matter, and the process by which this quantity is evaluated. This quantity is the limiting ratio of the quantity of the coloring matter in an element of space to the volume of that element. Now if we should take for our elements of volume, after any amount of stirring, the spaces occupied by the same portions of the liquid which originally occupied any given system of elements of volume, the densities of the coloring matter, thus estimated, would be identical with the original densities as determined by the given system of elements of volume. Moreover, if at the end of any finite amount of stirring we should take our elements of volume in any ordinary form but sufficiently small, the average square of the density of the coloring matter, as determined by such element of volume, would approximate to any required degree to its value before the stirring. But if we take any element of space of fixed position and dimensions, we may continue the stirring so long that the densities of the colored liquid estimated for these fixed elements will approach a uniform limit, viz., that of perfect mixture.

The case is evidently one of those in which the limit of a limit has different values, according to the order in which we

apply the processes of taking a limit. If treating the elements of volume as constant, we continue the stirring indefinitely, we get a uniform density, a result not affected by making the elements as small as we choose; but if treating the amount of stirring as finite, we diminish indefinitely the elements of volume, we get exactly the same distribution in density as before the stirring, a result which is not affected by continuing the stirring as long as we choose. The question is largely one of language and definition. One may perhaps be allowed to say that a finite amount of stirring will not affect the mean square of the density of the coloring matter, but an infinite amount of stirring may be regarded as producing a condition in which the mean square of the density has its minimum value, and the density is uniform. We may certainly say that a sensibly uniform density of the colored component may be produced by stirring. Whether the time required for this result would be long or short depends upon the nature of the motion given to the liquid, and the fineness of our method of evaluating the density.

All this may appear more distinctly if we consider a special case of liquid motion. Let us imagine a cylindrical mass of liquid of which one sector of 90° is black and the rest white. Let it have a motion of rotation about the axis of the cylinder in which the angular velocity is a function of the distance from the axis. In the course of time the black and the white parts would become drawn out into thin ribbons, which would be wound spirally about the axis. The thickness of these ribbons would diminish without limit, and the liquid would therefore tend toward a state of perfect mixture of the black and white portions. That is, in any given element of space, the proportion of the black and white would approach 1 : 3 as a limit. Yet after any finite time, the total volume would be divided into two parts, one of which would consist of the white liquid exclusively, and the other of the black exclusively. If the coloring matter, instead of being distributed initially with a uniform density throughout a section of the cylinder, were distributed with a density represented by any arbitrary func-

tion of the cylindrical coördinates r, θ and z, the effect of the same motion continued indefinitely would be an approach to a condition in which the density is a function of r and z alone. In this limiting condition, the average square of the density would be less than in the original condition, when the density was supposed to vary with $\dot{c}$, although after any finite time the average square of the density would be the same as at first.

If we limit our attention to the motion in a single plane perpendicular to the axis of the cylinder, we have something which is almost identical with a diagrammatic representation of the changes in distribution in phase of an ensemble of systems of one degree of freedom, in which the motion is periodic, the period varying with the energy, as in the case of a pendulum swinging in a circular arc. If the coördinates and momenta of the systems are represented by rectangular coördinates in the diagram, the points in the diagram representing the changing phases of moving systems, will move about the origin in closed curves of constant energy. The motion will be such that areas bounded by points representing moving systems will be preserved. The only difference between the motion of the liquid and the motion in the diagram is that in one case the paths are circular, and in the other they differ more or less from that form.

When the energy is proportional to $p^2 + q^2$ the curves of constant energy are circles, and the period is independent of the energy. There is then no tendency toward a state of statistical equilibrium. The diagram turns about the origin without change of form. This corresponds to the case of liquid motion, when the liquid revolves with a uniform angular velocity like a rigid solid.

The analogy between the motion of an ensemble of systems in an extension-in-phase and a steady current in an incompressible liquid, and the diagrammatic representation of the case of one degree of freedom, which appeals to our geometrical intuitions, may be sufficient to show how the conservation of density in phase, which involves the conservation of the

average value of the index of probability of phase, is consistent with an approach to a limiting condition in which that average value is less. We might perhaps fairly infer from such considerations as have been adduced that an approach to a limiting condition of statistical equilibrium is the general rule, when the initial condition is not of that character. But the subject is of such importance that it seems desirable to give it farther consideration.

Let us suppose that the total extension-in-phase for the kind of system considered to be divided into equal elements (DV) which are very small but not infinitely small. Let us imagine an ensemble of systems distributed in this extension in a manner represented by the index of probability η, which is an arbitrary function of the phase subject only to the restriction expressed by equation (46) of Chapter I. We shall suppose the elements DV to be so small that η may in general be regarded as sensibly constant within any one of them at the initial moment. Let the path of a system be defined as the series of phases through which it passes.

At the initial moment (t') a certain system is in an element of extension DV'. Subsequently, at the time t'', the same system is in the element DV''. Other systems which were at first in DV' will at the time t'' be in DV'', but not all, probably. The systems which were at first in DV' will at the time t'' occupy an extension-in-phase exactly as large as at first. But it will probably be distributed among a very great number of the elements (DV) into which we have divided the total extension-in-phase. If it is not so, we can generally take a later time at which it will be so. There will be exceptions to this for particular laws of motion, but we will confine ourselves to what may fairly be called the general case. Only a very small part of the systems initially in DV' will be found in DV'' at the time t'', and those which are found in DV'' at that time were at the initial moment distributed among a very large number of elements DV.

What is important for our purpose is the value of η, the index of probability of phase in the element DV'' at the time

t''. In the part of DV'' occupied by systems which at the time t' were in DV' the value of η will be the same as its value in DV' at the time t', which we shall call η'. In the parts of DV'' occupied by systems which at t' were in elements very near to DV' we may suppose the value of η to vary little from η'. We cannot assume this in regard to parts of DV'' occupied by systems which at t' were in elements remote from DV'. We want, therefore, some idea of the nature of the extension-in-phase occupied at t' by the systems which at t'' will occupy DV''. Analytically, the problem is identical with finding the extension occupied at t'' by the systems which at t' occupied DV'. Now the systems in DV'' which lie on the same *path* as the system first considered, evidently arrived at DV'' at nearly the same time, and must have left DV' at nearly the same time, and therefore at t' were in or near DV'. We may therefore take η' as the value for these systems. The same essentially is true of systems in DV'' which lie on paths very close to the path already considered. But with respect to paths passing through DV' and DV'', but not so close to the first path, we cannot assume that the time required to pass from DV' to DV'' is nearly the same as for the first path. The difference of the times required may be small in comparison with $t''-t'$, but as this interval can be as large as we choose, the difference of the times required in the different paths has no limit to its possible value. Now if the case were one of statistical equilibrium, the value of η would be constant in any path, and if all the paths which pass through DV'' also pass through or near DV', the value of η throughout DV'' will vary little from η'. But when the case is not one of statistical equilibrium, we cannot draw any such conclusion. The only conclusion which we can draw with respect to the phase at t' of the systems which at t'' are in DV'' is that they are nearly on the same path.

Now if we should make a new estimate of indices of probability of phase at the time t'', using for this purpose the elements DV, — that is, if we should divide the number of

systems in DV'', for example, by the total number of systems, and also by the extension-in-phase of the element, and take the logarithm of the quotient, we would get a number which would be less than the average value of η for the systems within DV'' based on the distribution in phase at the time t'.* Hence the average value of η for the whole ensemble of systems based on the distribution at t'' will be less than the average value based on the distribution at t'.

We must not forget that there are exceptions to this general rule. These exceptions are in cases in which the laws of motion are such that systems having small differences of phase will continue always to have small differences of phase.

It is to be observed that if the average index of probability in an ensemble may be said in some sense to have a less value at one time than at another, it is not necessarily priority in time which determines the greater average index. If a distribution, which is not one of statistical equilibrium, should be given for a time t', and the distribution at an earlier time t'' should be defined as that given by the corresponding phases, if we increase the interval leaving t' fixed and taking t'' at an earlier and earlier date, the distribution at t'' will in general approach a limiting distribution which is in statistical equilibrium. The determining difference in such cases is that between a definite distribution at a definite time and the limit of a varying distribution when the moment considered is carried either forward or backward indefinitely.†

But while the distinction of prior and subsequent events may be immaterial with respect to mathematical fictions, it is quite otherwise with respect to the events of the real world. It should not be forgotten, when our ensembles are chosen to illustrate the probabilities of events in the real world, that

* See Chapter XI, Theorem IX.

† One may compare the kinematical truism that when two points are moving with uniform velocities, (with the single exception of the case where the relative motion is zero,) their mutual distance at any definite time is less than for $t = \infty$, or $t = -\infty$.

while the probabilities of subsequent events may often be determined from the probabilities of prior events, it is rarely the case that probabilities of prior events can be determined from those of subsequent events, for we are rarely justified in excluding the consideration of the antecedent probability of the prior events.

It is worthy of notice that to take a system at random from an ensemble at a date chosen at random from several given dates, t', t'', etc., is practically the same thing as to take a system at random from the ensemble composed of all the systems of the given ensemble in their phases at the time t', together with the same systems in their phases at the time t'', etc. By Theorem VIII of Chapter XI this will give an ensemble in which the average index of probability will be less than in the given ensemble, except in the case when the distribution in the given ensemble is the same at the times t', t'', etc. Consequently, any indefiniteness in the time in which we take a system at random from an ensemble has the practical effect of diminishing the average index of the ensemble from which the system may be supposed to be drawn, except when the given ensemble is in statistical equilibrium.

CHAPTER XIII.

EFFECT OF VARIOUS PROCESSES ON AN ENSEMBLE OF SYSTEMS.

In the last chapter and in Chapter I we have considered the changes which take place in the course of time in an ensemble of isolated systems. Let us now proceed to consider the changes which will take place in an ensemble of systems under external influences. These external influences will be of two kinds, the variation of the coördinates which we have called *external*, and the action of other ensembles of systems. The essential difference of the two kinds of influence consists in this, that the bodies to which the external coördinates relate are not distributed in phase, while in the case of interaction of the systems of two ensembles, we have to regard the fact that both are distributed in phase. To find the effect produced on the ensemble with which we are principally concerned, we have therefore to consider single values of what we have called external coördinates, but an infinity of values of the internal coördinates of any other ensemble with which there is interaction.

Or, — to regard the subject from another point of view, — the action between an unspecified system of an ensemble and the bodies represented by the external coördinates, is the action between a system imperfectly determined with respect to phase and one which is perfectly determined; while the interaction between two unspecified systems belonging to different ensembles is the action between two systems both of which are imperfectly determined with respect to phase.*

We shall suppose the ensembles which we consider to be distributed in phase in the manner described in Chapter I, and

* In the development of the subject, we shall find that this distinction corresponds to the distinction in thermodynamics between mechanical and thermal action.

represented by the notations of that chapter, especially by the index of probability of phase (η). There are therefore $2n$ independent variations in the phases which constitute the ensembles considered. This excludes ensembles like the microcanonical, in which, as energy is constant, there are only $2n - 1$ independent variations of phase. This seems necessary for the purposes of a general discussion. For although we may imagine a microcanonical ensemble to have a permanent existence when isolated from external influences, the effect of such influences would generally be to destroy the uniformity of energy in the ensemble. Moreover, since the microcanonical ensemble may be regarded as a limiting case of such ensembles as are described in Chapter I, (and that in more than one way, as shown in Chapter X,) the exclusion is rather formal than real, since any properties which belong to the microcanonical ensemble could easily be derived from those of the ensembles of Chapter I, which in a certain sense may be regarded as representing the general case.

Let us first consider the effect of variation of the external coördinates. We have already had occasion to regard these quantities as variable in the differentiation of certain equations relating to ensembles distributed according to certain laws called canonical or microcanonical. That variation of the external coördinates was, however, only carrying the attention of the mind from an ensemble with certain values of the external coördinates, and distributed in phase according to some general law depending upon those values, to another ensemble with different values of the external coördinates, and with the distribution changed to conform to these new values.

What we have now to consider is the effect which would actually result in the course of time in an ensemble of systems in which the external coördinates should be varied in any arbitrary manner. Let us suppose, in the first place, that these coördinates are varied abruptly at a given instant, being constant both before and after that instant. By the definition of the external coördinates it appears that this variation does not affect the phase of any system of the ensemble at the time

when it takes place. Therefore it does not affect the index of probability of phase (η) of any system, or the average value of the index ($\bar{\eta}$) at that time. And if these quantities are constant in time before the variation of the external coördinates, and after that variation, their constancy in time is not interrupted by that variation. In fact, in the demonstration of the conservation of probability of phase in Chapter I, the variation of the external coördinates was not excluded.

But a variation of the external coördinates will in general disturb a previously existing state of statistical equilibrium. For, although it does not affect (at the first instant) the distribution-in-phase, it does affect the condition necessary for equilibrium. This condition, as we have seen in Chapter IV, is that the index of probability of phase shall be a function of phase which is constant in time for moving systems. Now a change in the external coördinates, by changing the forces which act on the systems, will change the nature of the functions of phase which are constant in time. Therefore, the distribution in phase which was one of statistical equilibrium for the old values of the external coördinates, will not be such for the new values.

Now we have seen, in the last chapter, that when the distribution-in-phase is not one of statistical equilibrium, an ensemble of systems may, and in general will, after a longer or shorter time, come to a state which may be regarded, if very small differences of phase are neglected, as one of statistical equilibrium, and in which consequently the average value of the index ($\bar{\eta}$) is less than at first. It is evident, therefore, that a variation of the external coördinates, by disturbing a state of statistical equilibrium, may indirectly cause a diminution, (in a certain sense at least,) of the value of $\bar{\eta}$.

But if the change in the external coördinates is very small, the change in the distribution necessary for equilibrium will in general be correspondingly small. Hence, the original distribution in phase, since it differs little from one which would be in statistical equilibrium with the new values of the external coördinates, may be supposed to have a value of $\bar{\eta}$

which differs by a small quantity of the second order from the minimum value which characterizes the state of statistical equilibrium. And the diminution in the average index resulting in the course of time from the very small change in the external coördinates, cannot exceed this small quantity of the second order.

Hence also, if the change in the external coördinates of an ensemble initially in statistical equilibrium consists in successive very small changes separated by very long intervals of time in which the disturbance of statistical equilibrium becomes sensibly effaced, the final diminution in the average index of probability will in general be negligible, although the total change in the external coördinates is large. The result will be the same if the change in the external coördinates takes place continuously but sufficiently slowly.

Even in cases in which there is no tendency toward the restoration of statistical equilibrium in the lapse of time, a variation of external coördinates which would cause, if it took place in a short time, a great disturbance of a previous state of equilibrium, may, if sufficiently distributed in time, produce no sensible disturbance of the statistical equilibrium.

Thus, in the case of three degrees of freedom, let the systems be heavy points suspended by elastic massless cords, and let the ensemble be distributed in phase with a density proportioned to some function of the energy, and therefore in statistical equilibrium. For a change in the external coördinates, we may take a horizontal motion of the point of suspension. If this is moved a given distance, the resulting disturbance of the statistical equilibrium may evidently be diminished indefinitely by diminishing the velocity of the point of suspension. This will be true if the law of elasticity of the string is such that the period of vibration is independent of the energy, in which case there is no tendency in the course of time toward a state of statistical equilibrium, as well as in the more general case, in which there is a tendency toward statistical equilibrium.

That something of this kind will be true in general, the following considerations will tend to show.

We define a path as the series of phases through which a system passes in the course of time when the external coördinates have fixed values. When the external coördinates are varied, paths are changed. The path of a phase is the path to which that phase belongs. With reference to any ensemble of systems we shall denote by $\overline{D}|_p$ the average value of the density-in-phase in a path. This implies that we have a measure for comparing different portions of the path. We shall suppose the time required to traverse any portion of a path to be its measure for the purpose of determining this average.

With this understanding, let us suppose that a certain ensemble is in statistical equilibrium. In every element of extension-in-phase, therefore, the density-in-phase D is equal to its path-average $\overline{D}|_p$. Let a sudden small change be made in the external coördinates. The statistical equilibrium will be disturbed and we shall no longer have $D = \overline{D}|_p$ everywhere. This is not because D is changed, but because $\overline{D}|_p$ is changed, the paths being changed. It is evident that if $D > \overline{D}|_p$ in a part of a path, we shall have $D < \overline{D}|_p$ in other parts of the same path.

Now, if we should imagine a further change in the external coördinates of the same kind, we should expect it to produce an effect of the same kind. But the manner in which the second effect will be superposed on the first will be different, according as it occurs immediately after the first change or after an interval of time. If it occurs immediately after the first change, then in any element of phase in which the first change produced a positive value of $D - \overline{D}|_p$ the second change will add a positive value to the first positive value, and where $D - \overline{D}|_p$ was negative, the second change will add a negative value to the first negative value.

But if we wait a sufficient time before making the second change in the external coördinates, so that systems have passed from elements of phase in which $D - \overline{D}|_p$ was originally positive to elements in which it was originally negative, and *vice versa*, (the systems carrying with them the values

of $D - \overline{D}|_p$,) the positive values of $D - \overline{D}|_p$ caused by the second change will be in part superposed on negative values due to the first change, and *vice versa*.

The disturbance of statistical equilibrium, therefore, produced by a given change in the values of the external coördinates may be very much diminished by dividing the change into two parts separated by a sufficient interval of time, and a sufficient interval of time for this purpose is one in which the phases of the individual systems are entirely unlike the first, so that any individual system is differently affected by the change, although the whole ensemble is affected in nearly the same way. Since there is no limit to the diminution of the disturbance of equilibrium by division of the change in the external coördinates, we may suppose as a general rule that by diminishing the velocity of the changes in the external coördinates, a given change may be made to produce a very small disturbance of statistical equilibrium.

If we write $\bar{\eta}'$ for the value of the average index of probability before the variation of the external coördinates, and $\bar{\eta}''$ for the value after this variation, we shall have in any case

$$\bar{\eta}'' \leqq \bar{\eta}'$$

as the simple result of the variation of the external coördinates. This may be compared with the thermodynamic theorem that the entropy of a body cannot be diminished by mechanical (as distinguished from thermal) action.*

If we have (approximate) statistical equilibrium between the times t' and t'' (corresponding to $\bar{\eta}'$ and $\bar{\eta}''$), we shall have approximately

$$\bar{\eta}' = \bar{\eta}'',$$

which may be compared with the thermodynamic theorem that the entropy of a body is not (sensibly) affected by mechanical action, during which the body is at each instant (sensibly) in a state of thermodynamic equilibrium.

Approximate statistical equilibrium may usually be attained

* The correspondences to which the reader's attention is called are between $-\eta$ and entropy, and between Θ and temperature.

by a sufficiently slow variation of the external coördinates, just as approximate thermodynamic equilibrium may usually be attained by sufficient slowness in the mechanical operations to which the body is subject.

We now pass to the consideration of the effect on an ensemble of systems which is produced by the action of other ensembles with which it is brought into dynamical connection. In a previous chapter * we have imagined a dynamical connection arbitrarily created between the systems of two ensembles. We shall now regard the action between the systems of the two ensembles as a result of the variation of the external coördinates, which causes such variations of the internal coördinates as to bring the systems of the two ensembles within the range of each other's action.

Initially, we suppose that we have two separate ensembles of systems, E_1 and E_2. The numbers of degrees of freedom of the systems in the two ensembles will be denoted by n_1 and n_2 respectively, and the probability-coefficients by e^{η_1} and e^{η_2}. Now we may regard any system of the first ensemble combined with any system of the second as forming a single system of $n_1 + n_2$ degrees of freedom. Let us consider the ensemble (E_{12}) obtained by thus combining each system of the first ensemble with each of the second.

At the initial moment, which may be specified by a single accent, the probability-coefficient of any phase of the combined systems is evidently the product of the probability-coefficients of the phases of which it is made up. This may be expressed by the equation,

$$e^{\eta_{12}'} = e^{\eta_1'}\, e^{\eta_2'}, \tag{455}$$

or

$$\eta_{12}' = \eta_1' + \eta_2', \tag{456}$$

which gives

$$\bar{\eta}_{12}' = \bar{\eta}_1' + \bar{\eta}_2'. \tag{457}$$

The forces tending to vary the internal coördinates of the combined systems, together with those exerted by either system upon the bodies represented by the coördinates called

* See Chapter IV, page 37.

external, may be derived from a single force-function, which, taken negatively, we shall call the potential energy of the combined systems and denote by ϵ_{12}. But we suppose that initially none of the systems of the two ensembles E_1 and E_2 come within range of each other's action, so that the potential energy of the combined system falls into two parts relating separately to the systems which are combined. The same is obviously true of the kinetic energy of the combined compound system, and therefore of its total energy. This may be expressed by the equation

$$\epsilon_{12}' = \epsilon_1' + \epsilon_2', \tag{458}$$

which gives

$$\bar{\epsilon}_{12}' = \bar{\epsilon}_1' + \bar{\epsilon}_2'. \tag{459}$$

Let us now suppose that in the course of time, owing to the motion of the bodies represented by the coördinates called external, the forces acting on the systems and consequently their positions are so altered, that the systems of the ensembles E_1 and E_2 are brought within range of each other's action, and after such mutual influence has lasted for a time, by a further change in the external coördinates, perhaps a return to their original values, the systems of the two original ensembles are brought again out of range of each other's action. Finally, then, at a time specified by double accents, we shall have as at first

$$\bar{\epsilon}_{12}'' = \bar{\epsilon}_1'' + \bar{\epsilon}_2''. \tag{460}$$

But for the indices of probability we must write *

$$\bar{\eta}_1'' + \bar{\eta}_2'' \leqq \bar{\eta}_{12}''. \tag{461}$$

The considerations adduced in the last chapter show that it is safe to write

$$\bar{\eta}_{12}'' \leqq \bar{\eta}_{12}'. \tag{462}$$

We have therefore

$$\bar{\eta}_1'' + \bar{\eta}_2'' \leqq \bar{\eta}_1' + \bar{\eta}_2', \tag{463}$$

which may be compared with the thermodynamic theorem that

* See Chapter XI, Theorem VII.

the thermal contact of two bodies may increase but cannot diminish the sum of their entropies.

Let us especially consider the case in which the two original ensembles were both canonically distributed in phase with the respective moduli Θ_1 and Θ_2. We have then, by Theorem III of Chapter XI,

$$\bar{\eta}_1' + \frac{\bar{\epsilon}_1'}{\Theta_1} \leqq \bar{\eta}_1'' + \frac{\bar{\epsilon}_1''}{\Theta_1} \tag{464}$$

$$\bar{\eta}_2' + \frac{\bar{\epsilon}_2'}{\Theta_2} \leqq \bar{\eta}_2'' + \frac{\bar{\epsilon}_2''}{\Theta_2} \tag{465}$$

Whence with (463) we have

$$\frac{\bar{\epsilon}_1'}{\Theta_1} + \frac{\bar{\epsilon}_2'}{\Theta_2} \leqq \frac{\epsilon_1''}{\Theta_1} + \frac{\bar{\epsilon}_2''}{\Theta_2} \tag{466}$$

or

$$\frac{\bar{\epsilon}_1'' - \bar{\epsilon}_1'}{\Theta_1} + \frac{\bar{\epsilon}_2'' - \bar{\epsilon}_2'}{\Theta_2} \geqq 0. \tag{467}$$

If we write $\overline{W}$ for the average work done by the combined systems on the external bodies, we have by the principle of the conservation of energy

$$\overline{W} = \bar{\epsilon}_{12}' - \bar{\epsilon}_{12}'' = \bar{\epsilon}_1' - \bar{\epsilon}_1'' + \bar{\epsilon}_2' - \bar{\epsilon}_2''. \tag{468}$$

Now if $\overline{W}$ is negligible, we have

$$\bar{\epsilon}_1'' - \bar{\epsilon}_1' = -(\bar{\epsilon}_2'' - \bar{\epsilon}_2') \tag{469}$$

and (467) shows that the ensemble which has the greater modulus must lose energy. This result may be compared to the thermodynamic principle, that when two bodies of different temperatures are brought together, that which has the higher temperature will lose energy.

Let us next suppose that the ensemble E_2 is originally canonically distributed with the modulus Θ_2, but leave the distribution of the other arbitrary. We have, to determine the result of a similar process,

$$\bar{\eta}_1'' + \bar{\eta}_2'' \leqq \bar{\eta}_1' + \bar{\eta}_2'$$

$$\bar{\eta}_2' + \frac{\bar{\epsilon}_2'}{\Theta_2} \leqq \bar{\eta}_2'' + \frac{\bar{\epsilon}_2''}{\Theta_2}$$

Hence
$$\bar{\eta}_1'' + \frac{\bar{\epsilon}_2'}{\Theta_2} \leqq \bar{\eta}_1' + \frac{\bar{\epsilon}_2''}{\Theta_2} \tag{470}$$

which may be written

$$\bar{\eta}_1' - \bar{\eta}_1'' \geqq \frac{\bar{\epsilon}_2' - \bar{\epsilon}_2''}{\Theta_2} \tag{471}$$

This may be compared with the thermodynamic principle that when a body (which need not be in thermal equilibrium) is brought into thermal contact with another of a given temperature, the increase of entropy of the first cannot be less (algebraically) than the loss of heat by the second divided by its temperature. Where $\overline{W}$ is negligible, we may write

$$\bar{\eta}_1'' + \frac{\bar{\epsilon}_1''}{\Theta_2} \leqq \bar{\eta}_1' + \frac{\bar{\epsilon}_1'}{\Theta_2} \tag{472}$$

Now, by Theorem III of Chapter XI, the quantity

$$\bar{\eta}_1 + \frac{\bar{\epsilon}_1}{\Theta_2} \tag{473}$$

has a minimum value when the ensemble to which $\bar{\eta}_1$ and $\bar{\epsilon}_1$ relate is distributed canonically with the modulus Θ_2. If the ensemble had originally this distribution, the sign $<$ in (472) would be impossible. In fact, in this case, it would be easy to show that the preceding formulae on which (472) is founded would all have the sign $=$. But when the two ensembles are not both originally distributed canonically with the same modulus, the formulae indicate that the quantity (473) may be diminished by bringing the ensemble to which ϵ_1 and η_1 relate into connection with another which is canonically distributed with modulus Θ_2, and therefore, that by repeated operations of this kind the ensemble of which the original distribution was entirely arbitrary might be brought approximately into a state of canonical distribution with the modulus Θ_2. We may compare this with the thermodynamic principle that a body of which the original thermal state may be entirely arbitrary, may be brought approximately into a state of thermal equilibrium with any given temperature by repeated connections with other bodies of that temperature.

Let us now suppose that we have a certain number of ensembles, E_0, E_1, E_2, etc., distributed canonically with the respective moduli Θ_0, Θ_1, Θ_2, etc. By variation of the external coördinates of the ensemble E_0, let it be brought into connection with E_1, and then let the connection be broken. Let it then be brought into connection with E_2, and then let that connection be broken. Let this process be continued with respect to the remaining ensembles. We do not make the assumption, as in some cases before, that the work connected with the variation of the external coördinates is a negligible quantity. On the contrary, we wish especially to consider the case in which it is large. In the final state of the ensemble E_0, let us suppose that the external coördinates have been brought back to their original values, and that the average energy ($\bar{\epsilon}_0$) is the same as at first.

In our usual notations, using one and two accents to distinguish original and final values, we get by repeated applications of the principle expressed in (463)

$$\bar{\eta}_0' + \bar{\eta}_1' + \bar{\eta}_2' + \text{etc.} \geqq \bar{\eta}_0'' + \bar{\eta}_1'' + \bar{\eta}_2'' + \text{etc.} \tag{474}$$

But by Theorem III of Chapter XI,

$$\bar{\eta}_0'' + \frac{\bar{\epsilon}_0''}{\Theta_0} \geqq \bar{\eta}_0' + \frac{\bar{\epsilon}_0'}{\Theta_0}, \tag{475}$$

$$\bar{\eta}_1'' + \frac{\bar{\epsilon}_1''}{\Theta_1} \geqq \bar{\eta}_1' + \frac{\bar{\epsilon}_1'}{\Theta_1}, \tag{476}$$

$$\bar{\eta}_2'' + \frac{\bar{\epsilon}_2''}{\Theta_2} \geqq \bar{\eta}_2' + \frac{\bar{\epsilon}_2'}{\Theta_2}, \tag{477}$$

etc.

Hence
$$\frac{\bar{\epsilon}_0''}{\Theta_0} + \frac{\bar{\epsilon}_1''}{\Theta_1} + \frac{\bar{\epsilon}_2''}{\Theta_2} + \text{etc.} \geqq \frac{\bar{\epsilon}_0'}{\Theta_0} + \frac{\bar{\epsilon}_1'}{\Theta_1} + \frac{\bar{\epsilon}_2'}{\Theta_2} + \text{etc.} \tag{478}$$

or, since
$$\bar{\epsilon}_0' = \bar{\epsilon}_0'',$$

$$0 \geqq \frac{\bar{\epsilon}_1' - \bar{\epsilon}_1''}{\Theta_1} + \frac{\bar{\epsilon}_2' - \bar{\epsilon}_2''}{\Theta_2} + \text{etc.} \tag{479}$$

If we write $\overline{W}$ for the average work done on the bodies represented by the external coördinates, we have

$$\bar{\epsilon}_1' - \bar{\epsilon}_1'' + \bar{\epsilon}_2' - \bar{\epsilon}_2'' + \text{etc.} = \overline{W}. \tag{480}$$

If E_0, E_1, and E_2 are the only ensembles, we have

$$W \leqq \frac{\Theta_1 - \Theta_2}{\Theta_1} (\bar{\epsilon}_1' - \bar{\epsilon}_1''). \tag{481}$$

It will be observed that the relations expressed in the last three formulae between $\overline{W}$, $\bar{\epsilon}_1 - \bar{\epsilon}_1''$, $\bar{\epsilon}_2' - \bar{\epsilon}_2''$, etc., and Θ_1, Θ_2, etc. are precisely those which hold in a Carnot's cycle for the work obtained, the energy lost by the several bodies which serve as heaters or coolers, and their initial temperatures.

It will not escape the reader's notice, that while from one point of view the operations which are here described are quite beyond our powers of actual performance, on account of the impossibility of handling the immense number of systems which are involved, yet from another point of view the operations described are the most simple and accurate means of representing what actually takes place in our simplest experiments in thermodynamics. The states of the bodies which we handle are certainly not known to us exactly. What we know about a body can generally be described most accurately and most simply by saying that it is one taken at random from a great number (ensemble) of bodies which are completely described. If we bring it into connection with another body concerning which we have a similar limited knowledge, the state of the two bodies is properly described as that of a pair of bodies taken from a great number (ensemble) of pairs which are formed by combining each body of the first ensemble with each of the second.

Again, when we bring one body into thermal contact with another, for example, in a Carnot's cycle, when we bring a mass of fluid into thermal contact with some other body from which we wish it to receive heat, we may do it by moving the vessel containing the fluid. This motion is mathematically expressed by the variation of the coördinates which determine the position of the vessel. We allow ourselves for the purposes of a theoretical discussion to suppose that the walls of this vessel are incapable of absorbing heat from the fluid.

Yet while we exclude the kind of action which we call thermal between the fluid and the containing vessel, we allow the kind which we call work in the narrower sense, which takes place when the volume of the fluid is changed by the motion of a piston. This agrees with what we have supposed in regard to the external coördinates, which we may vary in any arbitrary manner, and are in this entirely unlike the coordinates of the second ensemble with which we bring the first into connection.

When heat passes in any thermodynamic experiment between the fluid principally considered and some other body, it is actually absorbed and given out by the walls of the vessel, which will retain a varying quantity. This is, however, a disturbing circumstance, which we suppose in some way made negligible, and actually neglect in a theoretical discussion. In our case, we suppose the walls incapable of absorbing energy, except through the motion of the external coördinates, but that they allow the systems which they contain to act directly on one another. Properties of this kind are mathematically expressed by supposing that in the vicinity of a certain surface, the position of which is determined by certain (external) coördinates, particles belonging to the system in question experience a repulsion from the surface increasing so rapidly with nearness to the surface that an infinite expenditure of energy would be required to carry them through it. It is evident that two systems might be separated by a surface or surfaces exerting the proper forces, and yet approach each other closely enough to exert mechanical action on each other.

CHAPTER XIV.

DISCUSSION OF THERMODYNAMIC ANALOGIES.

If we wish to find in rational mechanics an *a priori* foundation for the principles of thermodynamics, we must seek mechanical definitions of temperature and entropy. The quantities thus defined must satisfy (under conditions and with limitations which again must be specified in the language of mechanics) the differential equation

$$d\epsilon = T\,d\eta - A_1\,da_1 - A_2\,da_2 - \text{etc.}, \qquad (482)$$

where ϵ, T, and η denote the energy, temperature, and entropy of the system considered, and $A_1\,da_1$, etc., the mechanical work (in the narrower sense in which the term is used in thermodynamics, *i. e.*, with exclusion of thermal action) done upon external bodies.

This implies that we are able to distinguish in mechanical terms the thermal action of one system on another from that which we call mechanical in the narrower sense, if not indeed in every case in which the two may be combined, at least so as to specify cases of thermal action and cases of mechanical action.

Such a differential equation moreover implies a finite equation between ϵ, η, and a_1, a_2, etc., which may be regarded as fundamental in regard to those properties of the system which we call thermodynamic, or which may be called so from analogy. This fundamental thermodynamic equation is determined by the fundamental mechanical equation which expresses the energy of the system as function of its momenta and coördinates with those external coördinates (a_1, a_2, etc.) which appear in the differential expression of the work done on external bodies. We have to show the mathematical operations by which the fundamental thermodynamic equation,

which in general is an equation of few variables, is derived from the fundamental mechanical equation, which in the case of the bodies of nature is one of an enormous number of variables.

We have also to enunciate in mechanical terms, and to prove, what we call the tendency of heat to pass from a system of higher temperature to one of lower, and to show that this tendency vanishes with respect to systems of the same temperature.

At least, we have to show by *a priori* reasoning that for such systems as the material bodies which nature presents to us, these relations hold with such approximation that they are sensibly true for human faculties of observation. This indeed is all that is really necessary to establish the science of thermodynamics on an *a priori* basis. Yet we will naturally desire to find the exact expression of those principles of which the laws of thermodynamics are the approximate expression. A very little study of the statistical properties of conservative systems of a finite number of degrees of freedom is sufficient to make it appear, more or less distinctly, that the general laws of thermodynamics are the limit toward which the exact laws of such systems approximate, when their number of degrees of freedom is indefinitely increased. And the problem of finding the exact relations, as distinguished from the approximate, for systems of a great number of degrees of freedom, is practically the same as that of finding the relations which hold for any number of degrees of freedom, as distinguished from those which have been established on an empirical basis for systems of a great number of degrees of freedom.

The enunciation and proof of these exact laws, for systems of any finite number of degrees of freedom, has been a principal object of the preceding discussion. But it should be distinctly stated that, if the results obtained when the numbers of degrees of freedom are enormous coincide sensibly with the general laws of thermodynamics, however interesting and significant this coincidence may be, we are still far from

having explained the phenomena of nature with respect to these laws. For, as compared with the case of nature, the systems which we have considered are of an ideal simplicity. Although our only assumption is that we are considering conservative systems of a finite number of degrees of freedom, it would seem that this is assuming far too much, so far as the bodies of nature are concerned. The phenomena of radiant heat, which certainly should not be neglected in any complete system of thermodynamics, and the electrical phenomena associated with the combination of atoms, seem to show that the hypothesis of systems of a finite number of degrees of freedom is inadequate for the explanation of the properties of bodies.

Nor do the results of such assumptions in every detail appear to agree with experience. We should expect, for example, that a diatomic gas, so far as it could be treated independently of the phenomena of radiation, or of any sort of electrical manifestations, would have six degrees of freedom for each molecule. But the behavior of such a gas seems to indicate not more than five.

But although these difficulties, long recognized by physicists,* seem to prevent, in the present state of science, any satisfactory explanation of the phenomena of thermodynamics as presented to us in nature, the ideal case of systems of a finite number of degrees of freedom remains as a subject which is certainly not devoid of a theoretical interest, and which may serve to point the way to the solution of the far more difficult problems presented to us by nature. And if the study of the statistical properties of such systems gives us an exact expression of laws which in the limiting case take the form of the received laws of thermodynamics, its interest is so much the greater.

Now we have defined what we have called the *modulus* (Θ) of an ensemble of systems canonically distributed in phase, and what we have called the index of probability (η) of any phase in such an ensemble. It has been shown that between

* See Boltzmann, Sitzb. der Wiener Akad., Bd. LXIII., S. 418, (1871).

the modulus (Θ), the external coördinates (a_1, etc.), and the average values in the ensemble of the energy (ϵ), the index of probability (η), and the external forces (A_1, etc.) exerted by the systems, the following differential equation will hold:

$$d\bar{\epsilon} = -\Theta\, d\bar{\eta} - \bar{A}_1\, da_1 - \bar{A}_2\, da_2 - \text{etc.} \qquad (483)$$

This equation, if we neglect the sign of averages, is identical in form with the thermodynamic equation (482), the modulus (Θ) corresponding to temperature, and the index of probability of phase with its sign reversed corresponding to entropy.*

We have also shown that the average square of the anomalies of ϵ, that is, of the deviations of the individual values from the average, is in general of the same order of magnitude as the reciprocal of the number of degrees of freedom, and therefore to human observation the individual values are indistinguishable from the average values when the number of degrees of freedom is very great.† In this case also the anomalies of η are practically insensible. The same is true of the anomalies of the external forces (A_1, etc.), so far as these are the result of the anomalies of energy, so that when these forces are sensibly determined by the energy and the external coördinates, and the number of degrees of freedom is very great, the anomalies of these forces are insensible.

The mathematical operations by which the finite equation between $\bar{\epsilon}$, $\bar{\eta}$, and a_1, etc., is deduced from that which gives the energy (ϵ) of a system in terms of the momenta ($p_1 \ldots p_n$) and coördinates both internal ($q_1 \ldots q_n$) and external (a_1, etc.), are indicated by the equation

$$e^{-\frac{\psi}{\Theta}} = \int_{\text{phases}}^{\text{all}} \ldots \int e^{-\frac{\epsilon}{\Theta}} dq_1 \ldots dq_n dp_1 \ldots dp_n, \qquad (484)$$

where $$\psi = \Theta\bar{\eta} + \bar{\epsilon}.$$

We have also shown that when systems of different ensembles are brought into conditions analogous to thermal contact, the average result is a passage of energy from the ensemble

* See Chapter IV, pages 44, 45. † See Chapter VII, pages 73–75.

of the greater modulus to that of the less,* or in case of equal moduli, that we have a condition of statistical equilibrium in regard to the distribution of energy.†

Propositions have also been demonstrated analogous to those in thermodynamics relating to a Carnot's cycle,‡ or to the tendency of entropy to increase,§ especially when bodies of different temperature are brought into contact.‖

We have thus precisely defined quantities, and rigorously demonstrated propositions, which hold for any number of degrees of freedom, and which, when the number of degrees of freedom (n) is enormously great, would appear to human faculties as the quantities and propositions of empirical thermodynamics.

It is evident, however, that there may be more than one quantity defined for finite values of n, which approach the same limit, when n is increased indefinitely, and more than one proposition relating to finite values of n, which approach the same limiting form for $n = \infty$. There may be therefore, and there are, other quantities which may be thought to have some claim to be regarded as temperature and entropy with respect to systems of a finite number of degrees of freedom.

The definitions and propositions which we have been considering relate essentially to what we have called a canonical ensemble of systems. This may appear a less natural and simple conception than what we have called a microcanonical ensemble of systems, in which all have the same energy, and which in many cases represents simply the *time-ensemble*, or ensemble of phases through which a single system passes in the course of time.

It may therefore seem desirable to find definitions and propositions relating to these microcanonical ensembles, which shall correspond to what in thermodynamics are based on experience. Now the differential equation

$$d\epsilon = e^{-\phi} V\, d \log V - \overline{A_1}|_\epsilon\, da_1 - \overline{A_2}|_\epsilon\, da_2 - \text{etc.}, \qquad (485)$$

* See Chapter XIII, page 160.
† See Chapter IV, pages 35–37.
‡ See Chapter XIII, pages 162, 163.
§ See Chapter XII, pages 143–151.
‖ See Chapter XIII, page 159.

which has been demonstrated in Chapter X, and which relates to a microcanonical ensemble, $\overline{A_1}|_\epsilon$ denoting the average value of A_1 in such an ensemble, corresponds precisely to the thermodynamic equation, except for the sign of average applied to the external forces. But as these forces are not entirely determined by the energy with the external coördinates, the use of average values is entirely germane to the subject, and affords the readiest means of getting perfectly determined quantities. These averages, which are taken for a microcanonical ensemble, may seem from some points of view a more simple and natural conception than those which relate to a canonical ensemble. Moreover, the energy, and the quantity corresponding to entropy, are free from the sign of average in this equation.

The quantity in the equation which corresponds to entropy is $\log V$, the quantity V being defined as the extension-in-phase within which the energy is less than a certain limiting value (ϵ). This is certainly a more simple conception than the average value in a canonical ensemble of the index of probability of phase. $\log V$ has the property that when it is constant

$$d\epsilon = -\overline{A_1}|_\epsilon \, da_1 - \overline{A_2}|_\epsilon \, da_2 + \text{etc.}, \tag{486}$$

which closely corresponds to the thermodynamic property of entropy, that when it is constant

$$d\epsilon = -A_1 \, da_1 - A_2 \, da_2 + \text{etc.} \tag{487}$$

The quantity in the equation which corresponds to temperature is $\epsilon^{-\phi} V$, or $d\epsilon / d \log V$. In a canonical ensemble, the average value of this quantity is equal to the modulus, as has been shown by different methods in Chapters IX and X.

In Chapter X it has also been shown that if the systems of a microcanonical ensemble consist of parts with separate energies, the average value of $\epsilon^{-\phi} V$ for any part is equal to its average value for any other part, and to the uniform value of the same expression for the whole ensemble. This corresponds to the theorem in the theory of heat that in case of thermal equilibrium the temperatures of the parts of a body are equal to one another and to that of the whole body.

Since the energies of the parts of a body cannot be supposed to remain absolutely constant, even where this is the case with respect to the whole body, it is evident that if we regard the temperature as a function of the energy, the taking of average or of probable values, or some other statistical process, must be used with reference to the parts, in order to get a perfectly definite value corresponding to the notion of temperature.

It is worthy of notice in this connection that the average value of the kinetic energy, either in a microcanonical ensemble, or in a canonical, divided by one half the number of degrees of freedom, is equal to $\epsilon^{-\phi} V$, or to its average value, and that this is true not only of the whole system which is distributed either microcanonically or canonically, but also of any part, although the corresponding theorem relating to temperature hardly belongs to empirical thermodynamics, since neither the (inner) kinetic energy of a body, nor its number of degrees of freedom is immediately cognizable to our faculties, and we meet the gravest difficulties when we endeavor to apply the theorem to the theory of gases, except in the simplest case, that of the gases known as monatomic.

But the correspondence between $\epsilon^{-\phi} V$ or $d\epsilon/d \log V$ and temperature is imperfect. If two isolated systems have such energies that

$$\frac{d\epsilon_1}{d \log V_1} = \frac{d\epsilon_2}{d \log V_2},$$

and the two systems are regarded as combined to form a third system with energy

$$\epsilon_{12} = \epsilon_1 + \epsilon_2,$$

we shall not have in general

$$\frac{d\epsilon_{12}}{d \log V_{12}} = \frac{d\epsilon_1}{d \log V_1} = \frac{d\epsilon_2}{d \log V_2},$$

as analogy with temperature would require. In fact, we have seen that

$$\frac{d\epsilon_{12}}{d \log V_{12}} = \overline{\frac{d\epsilon_1}{d \log V_1}}\Big|_{\epsilon_{12}} = \overline{\frac{d\epsilon_2}{d \log V_2}}\Big|_{\epsilon_{12}},$$

where the second and third members of the equation denote average values in an ensemble in which the compound system is microcanonically distributed in phase. Let us suppose the two original systems to be identical in nature. Then

$$\epsilon_1 = \epsilon_2 = \overline{\epsilon_1}|_{\epsilon_{12}} = \overline{\epsilon_2}|_{\epsilon_{12}}.$$

The equation in question would require that

$$\frac{d\epsilon_1}{d \log V_1} = \overline{\frac{d\epsilon_1}{d \log V_1}}\Big|_{\epsilon_{12}},$$

i. e., that we get the same result, whether we take the value of $d\epsilon_1/d \log V_1$ determined for the average value of ϵ_1 in the ensemble, or take the average value of $d\epsilon_1/d \log V_1$. This will be the case where $d\epsilon_1/d \log V_1$ is a linear function of ϵ_1. Evidently this does not constitute the most general case. Therefore the equation in question cannot be true in general. It is true, however, in some very important particular cases, as when the energy is a quadratic function of the p's and q's, or of the p's alone.* When the equation holds, the case is analogous to that of bodies in thermodynamics for which the specific heat for constant volume is constant.

Another quantity which is closely related to temperature is $d\phi/d\epsilon$. It has been shown in Chapter IX that in a canonical ensemble, if $n > 2$, the average value of $d\phi/d\epsilon$ is $1/\Theta$, and that the most common value of the energy in the ensemble is that for which $d\phi/d\epsilon = 1/\Theta$. The first of these properties may be compared with that of $d\epsilon/d \log V$, which has been seen to have the average value Θ in a canonical ensemble, without restriction in regard to the number of degrees of freedom.

With respect to microcanonical ensembles also, $d\phi/d\epsilon$ has a property similar to what has been mentioned with respect to $d\epsilon/d \log V$. That is, if a system microcanonically distributed in phase consists of two parts with separate energies, and each

* This last case is important on account of its relation to the theory of gases, although it must in strictness be regarded as a limit of possible cases, rather than as a case which is itself possible.

with more than two degrees of freedom, the average values in the ensemble of $d\phi/d\epsilon$ for the two parts are equal to one another and to the value of same expression for the whole. In our usual notations

$$\overline{\frac{d\phi_1}{d\epsilon_1}}\Big|_{\epsilon_{12}} = \overline{\frac{d\phi_2}{d\epsilon_2}}\Big|_{\epsilon_{12}} = \frac{d\phi_{12}}{d\epsilon_{12}}$$

if $n_1 > 2$, and $n_2 > 2$.

This analogy with temperature has the same incompleteness which was noticed with respect to $d\epsilon/d\log V$, viz., if two systems have such energies (ϵ_1 and ϵ_2) that

$$\frac{d\phi_1}{d\epsilon_1} = \frac{d\phi_2}{d\epsilon_2},$$

and they are combined to form a third system with energy

$$\epsilon_{12} = \epsilon_1 + \epsilon_2,$$

we shall not have in general

$$\frac{d\phi_{12}}{d\epsilon_{12}} = \frac{d\phi_1}{d\epsilon_1} = \frac{d\phi_2}{d\epsilon_2}.$$

Thus, if the energy is a quadratic function of the p's and q's, we have *

$$\frac{d\phi_1}{d\epsilon_1} = \frac{n_1 - 1}{\epsilon_1}, \qquad \frac{d\phi_2}{d\epsilon_2} = \frac{n_2 - 1}{\epsilon_2},$$

$$\frac{d\phi_{12}}{d\epsilon_{12}} = \frac{n_{12} - 1}{\epsilon_{12}} = \frac{n_1 + n_2 - 1}{\epsilon_1 + \epsilon_2},$$

where n_1, n_2, n_{12}, are the numbers of degrees of freedom of the separate and combined systems. But

$$\frac{d\phi_1}{d\epsilon_1} = \frac{d\phi_2}{d\epsilon_2} = \frac{n_1 + n_2 - 2}{\epsilon_1 + \epsilon_2}.$$

If the energy is a quadratic function of the p's alone, the case would be the same except that we should have $\frac{1}{2}n_1$, $\frac{1}{2}n_2$, $\frac{1}{2}n_{12}$, instead of n_1, n_2, n_{12}. In these particular cases, the analogy

* See foot-note on page 93. We have here made the least value of the energy consistent with the values of the external coördinates zero instead of ϵ_a, as is evidently allowable when the external coördinates are supposed invariable.

between $d\epsilon/d\log V$ and temperature would be complete, as has already been remarked. We should have

$$\frac{d\epsilon_1}{d\log V_1}=\frac{\epsilon_1}{n_1},\qquad \frac{d\epsilon_2}{d\log V_2}=\frac{\epsilon_2}{n_2},$$

$$\frac{d\epsilon_{12}}{d\log V_{12}}=\frac{\epsilon_{12}}{n_{12}}=\frac{d\epsilon_1}{d\log V_1}=\frac{d\epsilon_2}{d\log V_2},$$

when the energy is a quadratic function of the p's and q's, and similar equations with $\frac{1}{2}n_1$, $\frac{1}{2}n_2$, $\frac{1}{2}n_{12}$, instead of n_1, n_2, n_{12}, when the energy is a quadratic function of the p's alone.

More characteristic of $d\phi/d\epsilon$ are its properties relating to most probable values of energy. If a system having two parts with separate energies and each with more than two degrees of freedom is microcanonically distributed in phase, the most probable division of energy between the parts, in a system taken at random from the ensemble, satisfies the equation

$$\frac{d\phi_1}{d\epsilon_1}=\frac{d\phi_2}{d\epsilon_2}, \tag{488}$$

which corresponds to the thermodynamic theorem that the distribution of energy between the parts of a system, in case of thermal equilibrium, is such that the temperatures of the parts are equal.

To prove the theorem, we observe that the fractional part of the whole number of systems which have the energy of one part (ϵ_1) between the limits ϵ_1' and ϵ_1'' is expressed by

$$e^{-\phi_{12}}\int_{\epsilon_1'}^{\epsilon_1''}e^{\phi_1+\phi_2}\,d\epsilon_1,$$

where the variables are connected by the equation

$$\epsilon_1+\epsilon_2=\text{constant}=\epsilon_{12}.$$

The greatest value of this expression, for a constant infinitesimal value of the difference $\epsilon_1''-\epsilon_1'$, determines a value of ϵ_1, which we may call its most probable value. This depends on the greatest possible value of $\phi_1+\phi_2$. Now if $n_1>2$, and $n_2>2$, we shall have $\phi_1=-\infty$ for the least possible value of

ϵ_1, and $\phi_2 = -\infty$ for the least possible value of ϵ_2. Between these limits ϕ_1 and ϕ_2 will be finite and continuous. Hence $\phi_1 + \phi_2$ will have a maximum satisfying the equation (488).

But if $n_1 \leqq 2$, or $n_2 \leqq 2$, $d\phi_1/d\epsilon_1$ or $d\phi_2/d\epsilon_2$ may be negative, or zero, for all values of ϵ_1 or ϵ_2, and can hardly be regarded as having properties analogous to temperature.

It is also worthy of notice that if a system which is microcanonically distributed in phase has three parts with separate energies, and each with more than two degrees of freedom, the most probable division of energy between these parts satisfies the equation

$$\frac{d\phi_1}{d\epsilon_1} = \frac{d\phi_2}{d\epsilon_2} = \frac{d\phi_3}{d\epsilon_3}.$$

That is, this equation gives the most probable set of values of ϵ_1, ϵ_2, and ϵ_3. But it does not give the most probable value of ϵ_1, or of ϵ_2, or of ϵ_3. Thus, if the energies are quadratic functions of the p's and q's, the most probable division of energy is given by the equation

$$\frac{n_1 - 1}{\epsilon_1} = \frac{n_2 - 1}{\epsilon_1} = \frac{n_3 - 1}{\epsilon_3}.$$

But the most probable value of ϵ_1 is given by

$$\frac{n_1 - 1}{\epsilon_1} = \frac{n_2 + n_3 - 1}{\epsilon_2 + \epsilon_3},$$

while the preceding equations give

$$\frac{n_1 - 1}{\epsilon_1} = \frac{n_2 + n_3 - 2}{\epsilon_2 + \epsilon_3}.$$

These distinctions vanish for very great values of n_1, n_2, n_3. For small values of these numbers, they are important. Such facts seem to indicate that the consideration of the most probable division of energy among the parts of a system does not afford a convenient foundation for the study of thermodynamic analogies in the case of systems of a small number of degrees of freedom. The fact that a certain division of energy is the most probable has really no especial physical importance, except when the ensemble of possible divisions are grouped so

closely together that the most probable division may fairly represent the whole. This is in general the case, to a very close approximation, when n is enormously great; it entirely fails when n is small.

If we regard $d\phi/d\epsilon$ as corresponding to the reciprocal of temperature, or, in other words, $d\epsilon/d\phi$ as corresponding to temperature, ϕ will correspond to entropy. It has been defined as $\log (dV/d\epsilon)$. In the considerations on which its definition is founded, it is therefore very similar to $\log V$. We have seen that $d\phi/d\log V$ approaches the value unity when n is very great.*

To form a differential equation on the model of the thermodynamic equation (482), in which $d\epsilon/d\phi$ shall take the place of temperature, and ϕ of entropy, we may write

$$d\epsilon = \left(\frac{d\epsilon}{d\phi}\right)_a d\phi + \left(\frac{d\epsilon}{da_1}\right)_{\phi,a} da_1 + \left(\frac{d\epsilon}{da_2}\right)_{\phi,a} da_2 + \text{etc.}, \quad (489)$$

or

$$d\phi = \frac{d\phi}{d\epsilon} d\epsilon + \frac{d\phi}{da_1} da_1 + \frac{d\phi}{da_2} da_2 + \text{etc.} \quad (490)$$

With respect to the differential coefficients in the last equation, which corresponds exactly to (482) solved with respect to $d\eta$, we have seen that their average values in a canonical ensemble are equal to $1/\Theta$, and the averages of A_1/Θ, A_2/Θ, etc.† We have also seen that $d\epsilon/d\phi$ (or $d\phi/d\epsilon$) has relations to the most probable values of energy in parts of a microcanonical ensemble. That $(d\epsilon/da_1)_{\phi,a}$, etc., have properties somewhat analogous, may be shown as follows.

In a physical experiment, we measure a force by balancing it against another. If we should ask what force applied to increase or diminish a_1 would balance the action of the systems, it would be one which varies with the different systems. But we may ask what single force will make a given value of a_1 the most probable, and we shall find that under certain conditions $(d\epsilon/da_1)_{\phi,a}$ represents that force.

* See Chapter X, pages 120, 121.

† See Chapter IX, equations (321), (327).

To make the problem definite, let us consider a system consisting of the original system together with another having the coördinates a_1, a_2, etc., and forces A_1', A_2', etc., tending to increase those coördinates. These are in addition to the forces A_1, A_2, etc., exerted by the original system, and are derived from a force-function $(-\epsilon_q')$ by the equations

$$A_1' = -\frac{d\epsilon_q'}{da_1}, \qquad A_2' = -\frac{d\epsilon_q'}{da_2}, \qquad \text{etc.}$$

For the energy of the whole system we may write

$$\mathrm{E} = \epsilon + \epsilon_q' + \tfrac{1}{2} m_1 \dot{a}_1^2 + \tfrac{1}{2} m_2 \dot{a}_2^2 + \text{etc.},$$

and for the extension-in-phase of the whole system within any limits

$$\int \ldots \int dp_1 \ldots dq_n \, da_1 \, m_1 \, d\dot{a}_1 \, da_2 \, m_2 \, d\dot{a}_2 \ldots$$

or
$$\int \ldots \int e^{\phi} \, d\epsilon \, da_1 \, m_1 \, d\dot{a}_1 \, da_2 \, m_2 \, d\dot{a}_2 \ldots,$$

or again
$$\int \ldots \int e^{\phi} \, d\mathrm{E} \, da_1 \, m_1 \, d\dot{a}_1 \, da_2 \, m_2 \, d\dot{a}_2 \ldots,$$

since $d\epsilon = d\mathrm{E}$, when a_1, $\dot{a}_1$, a_2, $\dot{a}_2$, etc., are constant. If the limits are expressed by E and $\mathrm{E} + d\mathrm{E}$, a_1 and $a_1 + da_1$, $\dot{a}_1$ and $\dot{a}_1 + d\dot{a}_1$, etc., the integral reduces to

$$e^{\phi} \, d\mathrm{E} \, da_1 \, m_1 \, d\dot{a}_1 \, da_2 \, m_2 \, d\dot{a}_2 \ldots$$

The values of a_1, $\dot{a}_1$, a_2, $\dot{a}_2$, etc., which make this expression a maximum for constant values of the energy of the whole system and of the differentials $d\mathrm{E}$, da_1, $d\dot{a}_1$, etc., are what may be called the most probable values of a_1, $\dot{a}_1$, etc., in an ensemble in which the whole system is distributed microcanonically. To determine these values we have

$$de^{\phi} = 0,$$

when
$$d(\epsilon + \epsilon_q' + \tfrac{1}{2} m \dot{a}_1^2 + \tfrac{1}{2} m_2 \dot{a}_2^2 + \text{etc.}) = 0.$$

That is,
$$d\phi = 0,$$

when

$$\left(\frac{d\epsilon}{d\phi}\right)_a d\phi + \left(\frac{d\epsilon}{da_1}\right)_{\phi,a} da_1 - A_1' da_1 + \text{etc.} + m_1 \dot{a}_1 d\dot{a}_1 + \text{etc.} = 0.$$

This requires $\dot{a}_1 = 0, \quad \dot{a}_2 = 0, \quad \text{etc.},$

and $$\left(\frac{d\epsilon}{da_1}\right)_{\phi,a} = A_1', \quad \left(\frac{d\epsilon}{da_2}\right)_{\phi,a} = A_2', \quad \text{etc.}$$

This shows that for any given values of E, a_1, a_2, etc. $\left(\frac{d\epsilon}{da_1}\right)_{\phi,a}$, $\left(\frac{d\epsilon}{da_2}\right)_{\phi,a}$, etc., represent the forces (in the generalized sense) which the external bodies would have to exert to make these values of a_1, a_2, etc., the most probable under the conditions specified. When the differences of the external forces which are exerted by the different systems are negligible, $-(d\epsilon/da_1)_{\phi,a}$, etc., represent these forces.

It is certainly in the quantities relating to a canonical ensemble, $\bar{\epsilon}$, Θ, $\bar{\eta}$, $\bar{A}_1$, etc., a_1, etc., that we find the most complete correspondence with the quantities of the thermodynamic equation (482). Yet the conception itself of the canonical ensemble may seem to some artificial, and hardly germane to a natural exposition of the subject; and the quantities ϵ, $\frac{d\epsilon}{d \log V}$, $\log V$, $\bar{A}_1]_\epsilon$, etc., a_1, etc., or ϵ, $\frac{d\epsilon}{d\phi}$, ϕ, $\left(\frac{d\epsilon}{da_1}\right)_{\phi,a}$, etc., a_1, etc., which are closely related to ensembles of constant energy, and to average and most probable values in such ensembles, and most of which are defined without reference to any ensemble, may appear the most natural analogues of the thermodynamic quantities.

In regard to the naturalness of seeking analogies with the thermodynamic behavior of bodies in canonical or microcanonical ensembles of systems, much will depend upon how we approach the subject, especially upon the question whether we regard energy or temperature as an independent variable.

It is very natural to take energy for an independent variable rather than temperature, because ordinary mechanics furnishes us with a perfectly defined conception of energy, whereas the idea of something relating to a mechanical system and corre-

sponding to temperature is a notion but vaguely defined. Now if the state of a system is given by its energy and the external coördinates, it is incompletely defined, although its partial definition is perfectly clear as far as it goes. The ensemble of phases microcanonically distributed, with the given values of the energy and the external coördinates, will represent the imperfectly defined system better than any other ensemble or single phase. When we approach the subject from this side, our theorems will naturally relate to average values, or most probable values, in such ensembles.

In this case, the choice between the variables of (485) or of (489) will be determined partly by the relative importance which is attached to average and probable values. It would seem that in general average values are the most important, and that they lend themselves better to analytical transformations. This consideration would give the preference to the system of variables in which $\log V$ is the analogue of entropy. Moreover, if we make ϕ the analogue of entropy, we are embarrassed by the necessity of making numerous exceptions for systems of one or two degrees of freedom.

On the other hand, the definition of ϕ may be regarded as a little more simple than that of $\log V$, and if our choice is determined by the simplicity of the definitions of the analogues of entropy and temperature, it would seem that the ϕ system should have the preference. In our definition of these quantities, V was defined first, and e^{ϕ} derived from V by differentiation. This gives the relation of the quantities in the most simple analytical form. Yet so far as the notions are concerned, it is perhaps more natural to regard V as derived from e^{ϕ} by integration. At all events, e^{ϕ} may be defined independently of V, and its definition may be regarded as more simple as not requiring the determination of the zero from which V is measured, which sometimes involves questions of a delicate nature. In fact, the quantity e^{ϕ} may exist, when the definition of V becomes illusory for practical purposes, as the integral by which it is determined becomes infinite.

The case is entirely different, when we regard the tempera-

ture as an independent variable, and we have to consider a system which is described as having a certain temperature and certain values for the external coördinates. Here also the state of the system is not completely defined, and will be better represented by an ensemble of phases than by any single phase. What is the nature of such an ensemble as will best represent the imperfectly defined state?

When we wish to give a body a certain temperature, we place it in a bath of the proper temperature, and when we regard what we call thermal equilibrium as established, we say that the body has the same temperature as the bath. Perhaps we place a second body of standard character, which we call a thermometer, in the bath, and say that the first body, the bath, and the thermometer, have all the same temperature.

But the body under such circumstances, as well as the bath, and the thermometer, even if they were entirely isolated from external influences (which it is convenient to suppose in a theoretical discussion), would be continually changing in phase, and in energy as well as in other respects, although our means of observation are not fine enough to perceive these variations.

The series of phases through which the whole system runs in the course of time may not be entirely determined by the energy, but may depend on the initial phase in other respects. In such cases the ensemble obtained by the microcanonical distribution of the whole system, which includes all possible time-ensembles combined in the proportion which seems least arbitrary, will represent better than any one time-ensemble the effect of the bath. Indeed a single time-ensemble, when it is not also a microcanonical ensemble, is too ill-defined a notion to serve the purposes of a general discussion. We will therefore direct our attention, when we suppose the body placed in a bath, to the microcanonical ensemble of phases thus obtained.

If we now suppose the quantity of the substance forming the bath to be increased, the anomalies of the separate energies of the body and of the thermometer in the microcanonical

ensemble will be increased, but not without limit. The anomalies of the energy of the bath, considered in comparison with its whole energy, diminish indefinitely as the quantity of the bath is increased, and become in a sense negligible, when the quantity of the bath is sufficiently increased. The ensemble of phases of the body, and of the thermometer, approach a standard form as the quantity of the bath is indefinitely increased. This limiting form is easily shown to be what we have described as the canonical distribution.

Let us write ϵ for the energy of the whole system consisting of the body first mentioned, the bath, and the thermometer (if any), and let us first suppose this system to be distributed canonically with the modulus Θ. We have by (205)

$$\overline{(\epsilon - \bar{\epsilon})^2} = \Theta^2 \frac{d\bar{\epsilon}}{d\Theta},$$

and since

$$\bar{\epsilon}_p = \frac{n}{2}\Theta,$$

$$\frac{d\bar{\epsilon}}{d\Theta} = \frac{n}{2}\frac{d\bar{\epsilon}}{d\bar{\epsilon}_p}.$$

If we write $\Delta\epsilon$ for the anomaly of mean square, we have

$$(\Delta\epsilon)^2 = \overline{(\epsilon - \bar{\epsilon})^2}.$$

If we set

$$\Delta\Theta = \frac{d\Theta}{d\bar{\epsilon}}\Delta\epsilon,$$

$\Delta\Theta$ will represent approximately the increase of Θ which would produce an increase in the average value of the energy equal to its anomaly of mean square. Now these equations give

$$(\Delta\Theta)^2 = \frac{2\Theta^2 d\bar{\epsilon}_p}{n\, d\bar{\epsilon}},$$

which shows that we may diminish $\Delta\Theta$ indefinitely by increasing the quantity of the bath.

Now our canonical ensemble consists of an infinity of microcanonical ensembles, which differ only in consequence of the different values of the energy which is constant in each. If we consider separately the phases of the first body which

occur in the canonical ensemble of the whole system, these phases will form a canonical ensemble of the same modulus. This canonical ensemble of phases of the first body will consist of parts which belong to the different microcanonical ensembles into which the canonical ensemble of the whole system is divided.

Let us now imagine that the modulus of the principal canonical ensemble is increased by $2\Delta\Theta$, and its average energy by $2\Delta\epsilon$. The modulus of the canonical ensemble of the phases of the first body considered separately will be increased by $2\Delta\Theta$. We may regard the infinity of microcanonical ensembles into which we have divided the principal canonical ensemble as each having its energy increased by $2\Delta\epsilon$. Let us see how the ensembles of phases of the first body contained in these microcanonical ensembles are affected. We may assume that they will all be affected in about the same way, as all the differences which come into account may be treated as small. Therefore, the canonical ensemble formed by taking them together will also be affected in the same way. But we know how this is affected. It is by the increase of its modulus by $2\Delta\Theta$, a quantity which vanishes when the quantity of the bath is indefinitely increased.

In the case of an infinite bath, therefore, the increase of the energy of one of the microcanonical ensembles by $2\Delta\epsilon$, produces a vanishing effect on the distribution in energy of the phases of the first body which it contains. But $2\Delta\epsilon$ is more than the average difference of energy between the microcanonical ensembles. The distribution in energy of these phases is therefore the same in the different microcanonical ensembles, and must therefore be canonical, like that of the ensemble which they form when taken together.*

* In order to appreciate the above reasoning, it should be understood that the differences of energy which occur in the canonical ensemble of phases of the first body are not here regarded as vanishing quantities. To fix one's ideas, one may imagine that he has the fineness of perception to make these differences seem large. The difference between the part of these phases which belong to one microcanonical ensemble of the whole system and the part which belongs to another would still be imperceptible, when the quantity of the bath is sufficiently increased.

As a general theorem, the conclusion may be expressed in the words: — If a system of a great number of degrees of freedom is microcanonically distributed in phase, any very small part of it may be regarded as canonically distributed.*

It would seem, therefore, that a canonical ensemble of phases is what best represents, with the precision necessary for exact mathematical reasoning, the notion of a body with a given temperature, if we conceive of the temperature as the state produced by such processes as we actually use in physics to produce a given temperature. Since the anomalies of the body increase with the quantity of the bath, we can only get rid of all that is arbitrary in the ensemble of phases which is to represent the notion of a body of a given temperature by making the bath infinite, which brings us to the canonical distribution.

A comparison of temperature and entropy with their analogues in statistical mechanics would be incomplete without a consideration of their differences with respect to units and zeros, and the numbers used for their numerical specification. If we apply the notions of statistical mechanics to such bodies as we usually consider in thermodynamics, for which the kinetic energy is of the same order of magnitude as the unit of energy, but the number of degrees of freedom is enormous, the values of Θ, $d\epsilon/d\log V$, and $d\epsilon/d\phi$ will be of the same order of magnitude as $1/n$, and the variable part of $\bar{\eta}$, $\log V$, and ϕ will be of the same order of magnitude as n.† If these quantities, therefore, represent in any sense the notions of temperature and entropy, they will nevertheless not be measured in units of the usual order of magnitude, — a fact which must be borne in mind in determining what magnitudes may be regarded as insensible to human observation.

Now nothing prevents our supposing energy and time in our statistical formulae to be measured in such units as may

* It is assumed — and without this assumption the theorem would have no distinct meaning — that the part of the ensemble considered may be regarded as having separate energy.

† See equations (124), (288), (289), and (314); also page 106.

be convenient for physical purposes. But when these units have been chosen, the numerical values of Θ, $d\epsilon/d\log V$, $d\epsilon/d\phi$, $\bar{\eta}$, $\log V$, ϕ, are entirely determined,* and in order to compare them with temperature and entropy, the numerical values of which depend upon an arbitrary unit, we must multiply all values of Θ, $d\epsilon/d\log V$, $d\epsilon/d\phi$ by a constant (K), and divide all values of $\bar{\eta}$, $\log V$, and ϕ by the same constant. This constant is the same for all bodies, and depends only on the units of temperature and energy which we employ. For ordinary units it is of the same order of magnitude as the numbers of atoms in ordinary bodies.

We are not able to determine the numerical value of K, as it depends on the number of molecules in the bodies with which we experiment. To fix our ideas, however, we may seek an expression for this value, based upon very probable assumptions, which will show how we would naturally proceed to its evaluation, if our powers of observation were fine enough to take cognizance of individual molecules.

If the unit of mass of a monatomic gas contains ν atoms, and it may be treated as a system of 3ν degrees of freedom, which seems to be the case, we have for canonical distribution

$$\bar{\epsilon}_p = \tfrac{3}{2}\nu\Theta,$$
$$\frac{d\bar{\epsilon}_p}{d\Theta} = \tfrac{3}{2}\nu. \qquad (491)$$

If we write T for temperature, and c_v for the specific heat of the gas for constant volume (or rather the limit toward which this specific heat tends, as rarefaction is indefinitely increased), we have

$$\frac{d\epsilon_p}{dT} = c_v, \qquad (492)$$

since we may regard the energy as entirely kinetic. We may set the ϵ_p of this equation equal to the $\bar{\epsilon}_p$ of the preceding,

* The unit of time only affects the last three quantities, and these only by an additive constant, which disappears (with the additive constant of entropy), when differences of entropy are compared with their statistical analogues. See page 19.

where indeed the individual values of which the average is taken would appear to human observation as identical. This gives

$$\frac{d\Theta}{dT} = \frac{2c_v}{3\nu},$$

whence

$$\frac{1}{K} = \frac{2c_v}{3\nu}. \tag{493}$$

a value recognized by physicists as a constant independent of the kind of monatomic gas considered.

We may also express the value of K in a somewhat different form, which corresponds to the indirect method by which physicists are accustomed to determine the quantity c_v. The kinetic energy due to the motions of the centers of mass of the molecules of a mass of gas sufficiently expanded is easily shown to be equal to

$$\tfrac{3}{2} p v,$$

where p and v denote the pressure and volume. The average value of the same energy in a canonical ensemble of such a mass of gas is

$$\tfrac{3}{2} \Theta \nu,$$

where ν denotes the number of molecules in the gas. Equating these values, we have

$$p v = \Theta \nu, \tag{494}$$

whence

$$\frac{1}{K} = \frac{\Theta}{T} = \frac{p v}{\nu T}. \tag{495}$$

Now the laws of Boyle, Charles, and Avogadro may be expressed by the equation

$$p v = A \nu T, \tag{496}$$

where A is a constant depending only on the units in which energy and temperature are measured. $1/K$, therefore, might be called the constant of the law of Boyle, Charles, and Avogadro as expressed with reference to the true number of molecules in a gaseous body.

Since such numbers are unknown to us, it is more convenient to express the law with reference to relative values. If we denote by M the so-called molecular weight of a gas, that

is, a number taken from a table of numbers proportional to the weights of various molecules and atoms, but having one of the values, perhaps the atomic weight of hydrogen, arbitrarily made unity, the law of Boyle, Charles, and Avogadro may be written in the more practical form

$$pv = A' T \frac{m}{M}, \tag{497}$$

where A' is a constant and m the weight of gas considered. It is evident that $\mathbf{1}\ K$ is equal to the product of the constant of the law in this form and the (true) weight of an atom of hydrogen, or such other atom or molecule as may be given the value unity in the table of molecular weights.

In the following chapter we shall consider the necessary modifications in the theory of equilibrium, when the quantity of matter contained in a system is to be regarded as variable, or, if the system contains more than one kind of matter, when the quantities of the several kinds of matter in the system are to be regarded as independently variable. This will give us yet another set of variables in the statistical equation, corresponding to those of the amplified form of the thermodynamic equation.

CHAPTER XV.

SYSTEMS COMPOSED OF MOLECULES.

THE nature of material bodies is such that especial interest attaches to the dynamics of systems composed of a great number of entirely similar particles, or, it may be, of a great number of particles of several kinds, all of each kind being entirely similar to each other. We shall therefore proceed to consider systems composed of such particles, whether in great numbers or otherwise, and especially to consider the statistical equilibrium of ensembles of such systems. One of the variations to be considered in regard to such systems is a variation in the numbers of the particles of the various kinds which it contains, and the question of statistical equilibrium between two ensembles of such systems relates in part to the tendencies of the various kinds of particles to pass from the one to the other.

First of all, we must define precisely what is meant by statistical equilibrium of such an ensemble of systems. The essence of statistical equilibrium is the permanence of the number of systems which fall within any given limits with respect to phase. We have therefore to define how the term "phase" is to be understood in such cases. If two phases differ only in that certain entirely similar particles have changed places with one another, are they to be regarded as identical or different phases? If the particles are regarded as indistinguishable, it seems in accordance with the spirit of the statistical method to regard the phases as identical. In fact, it might be urged that in such an ensemble of systems as we are considering no identity is possible between the particles of different systems except that of qualities, and if ν particles of one system are described as entirely similar to one another and to ν of another system, nothing remains on which to base

the indentification of any particular particle of the first system with any particular particle of the second. And this would be true, if the ensemble of systems had a simultaneous objective existence. But it hardly applies to the creations of the imagination. In the cases which we have been considering, and in those which we shall consider, it is not only possible to conceive of the motion of an ensemble of similar systems simply as possible cases of the motion of a single system, but it is actually in large measure for the sake of representing more clearly the possible cases of the motion of a single system that we use the conception of an ensemble of systems. The perfect similarity of several particles of a system will not in the least interfere with the identification of a particular particle in one case with a particular particle in another. The question is one to be decided in accordance with the requirements of practical convenience in the discussion of the problems with which we are engaged.

Our present purpose will often require us to use the terms *phase*, *density-in-phase*, *statistical equilibrium*, and other connected terms on the supposition that phases are *not* altered by the exchange of places between similar particles. Some of the most important questions with which we are concerned have reference to phases thus defined. We shall call them phases determined by generic definitions, or briefly, generic phases. But we shall also be obliged to discuss phases defined by the narrower definition (so that exchange of position between similar particles is regarded as changing the phase), which will be called phases determined by specific definitions, or briefly, specific phases. For the analytical description of a specific phase is more simple than that of a generic phase. And it is a more simple matter to make a multiple integral extend over all possible specific phases than to make one extend without repetition over all possible generic phases.

It is evident that if $\nu_1, \nu_2 \ldots \nu_h$, are the numbers of the different kinds of molecules in any system, the number of specific phases embraced in one generic phase is represented by the continued product $\lfloor\underline{\nu_1} \lfloor\underline{\nu_2} \ldots \lfloor\underline{\nu_h}$, and the coefficient of probabil-

ity of a generic phase is the sum of the probability-coefficients of the specific phases which it represents. When these are equal among themselves, the probability-coefficient of the generic phase is equal to that of the specific phase multiplied by $\lfloor\underline{\nu_1}\lfloor\underline{\nu_2}\ldots\lfloor\underline{\nu_h}$. It is also evident that statistical equilibrium may subsist with respect to generic phases without statistical equilibrium with respect to specific phases, but not *vice versa.*

Similar questions arise where one particle is capable of several equivalent positions. Does the change from one of these positions to another change the phase? It would be most natural and logical to make it affect the specific phase, but not the generic. The number of specific phases contained in a generic phase would then be $\lfloor\underline{\nu_1}\,\kappa_1^{\nu_1}\ldots\lfloor\underline{\nu_h}\,\kappa_h^{\nu_h}$, where $\kappa_1,\ldots\kappa_h$ denote the numbers of equivalent positions belonging to the several kinds of particles. The case in which a κ is infinite would then require especial attention. It does not appear that the resulting complications in the formulae would be compensated by any real advantage. The reason of this is that in problems of real interest equivalent positions of a particle will always be equally probable. In this respect, equivalent positions of the same particle are entirely unlike the $\lfloor\underline{\nu}$ different ways in which ν particles may be distributed in ν different positions. Let it therefore be understood that in spite of the physical equivalence of different positions of the same particle they are to be considered as constituting a difference of generic phase as well as of specific. The number of specific phases contained in a generic phase is therefore always given by the product $\lfloor\underline{\nu_1}\lfloor\underline{\nu_2}\ldots\lfloor\underline{\nu_h}$.

Instead of considering, as in the preceding chapters, ensembles of systems differing only in phase, we shall now suppose that the systems constituting an ensemble are composed of particles of various kinds, and that they differ not only in phase but also in the numbers of these particles which they contain. The external coördinates of all the systems in the ensemble are supposed, as heretofore, to have the same value, and when they vary, to vary together. For distinction, we may call such an ensemble a *grand ensemble*, and one in

which the systems differ only in phase a *petit ensemble.* A grand ensemble is therefore composed of a multitude of petit ensembles. The ensembles which we have hitherto discussed are petit ensembles.

Let $\nu_1, \ldots \nu_h$, etc., denote the numbers of the different kinds of particles in a system, ϵ its energy, and $q_1, \ldots q_n$, $p_1, \ldots p_n$ its coördinates and momenta. If the particles are of the nature of material points, the number of coördinates (n) of the system will be equal to $3\nu_1 \ldots + 3\nu_h$. But if the particles are less simple in their nature, if they are to be treated as rigid solids, the orientation of which must be regarded, or if they consist each of several atoms, so as to have more than three degrees of freedom, the number of coördinates of the system will be equal to the sum of ν_1, ν_2, etc., multiplied each by the number of degrees of freedom of the kind of particle to which it relates.

Let us consider an ensemble in which the number of systems having $\nu_1, \ldots \nu_h$ particles of the several kinds, and having values of their coördinates and momenta lying between the limits q_1 and $q_1 + dq_1$, p_1 and $p_1 + dp_1$, etc., is represented by the expression

$$\frac{N e^{\frac{\Omega+\mu_1\nu_1\ldots+\mu_h\nu_h-\epsilon}{\Theta}}}{\lfloor\nu_1 \ldots \lfloor\nu_h} dp_1 \ldots dq_n, \tag{498}$$

where $N, \Omega, \Theta, \mu_1, \ldots \mu_h$ are constants, N denoting the total number of systems in the ensemble. The expression

$$\frac{N e^{\frac{\Omega+\mu_1\nu_1\ldots+\mu_h\nu_h-\epsilon}{\Theta}}}{\lfloor\nu_1 \ldots \lfloor\nu_h} \tag{499}$$

evidently represents the density-in-phase of the ensemble within the limits described, that is, for a phase specifically defined. The expression

$$\frac{e^{\frac{\Omega+\mu_1\nu_1\ldots+\mu_h\nu_h-\epsilon}{\Theta}}}{\lfloor\nu_1 \ldots \lfloor\nu_h} \tag{500}$$

is therefore the probability-coefficient for a phase specifically defined. This has evidently the same value for all the $\lfloor\underline{\nu_1} \ldots \lfloor\underline{\nu_h}$ phases obtained by interchanging the phases of particles of the same kind. The probability-coefficient for a generic phase will be $\lfloor\underline{\nu_1} \ldots \lfloor\underline{\nu_h}$ times as great, viz.,

$$e^{\frac{\Omega+\mu_1\nu_1\ldots+\mu_h\nu_h-\epsilon}{\Theta}}. \tag{501}$$

We shall say that such an ensemble as has been described is *canonically distributed*, and shall call the constant Θ its modulus. It is evidently what we have called a grand ensemble. The petit ensembles of which it is composed are canonically distributed, according to the definitions of Chapter IV, since the expression

$$\frac{e^{\frac{\Omega+\mu_1\nu_1\ldots+\mu_h\nu_h}{\Theta}}}{\lfloor\underline{\nu_1} \ldots \lfloor\underline{\nu_h}} \tag{502}$$

is constant for each petit ensemble. The grand ensemble, therefore, is in statistical equilibrium with respect to specific phases.

If an ensemble, whether grand or petit, is identical so far as generic phases are concerned with one canonically distributed, we shall say that its distribution is canonical with respect to generic phases. Such an ensemble is evidently in statistical equilibrium with respect to generic phases, although it may not be so with respect to specific phases.

If we write H for the index of probability of a generic phase in a grand ensemble, we have for the case of canonical distribution

$$\mathrm{H} = \frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \epsilon}{\Theta}. \tag{503}$$

It will be observed that the H is a linear function of ϵ and $\nu_1, \ldots \nu_h$; also that whenever the index of probability of generic phases in a grand ensemble is a linear function of ϵ, $\nu_1, \ldots \nu_h$, the ensemble is canonically distributed with respect to generic phases.

The constant Ω we may regard as determined by the equation

$$N = \Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \int_{\text{phases}}^{\text{all}} \ldots \int \frac{Ne^{\frac{\Omega + \mu_1\nu_1 \ldots \mu_h\nu_h - \epsilon}{\Theta}}}{\lfloor\nu_1 \ldots \lfloor\nu_h} dp_1 \ldots dq_n, \qquad (504)$$

or

$$e^{-\frac{\Omega}{\Theta}} = \Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \frac{e^{\frac{\mu_1\nu_1 \ldots \mu_h\nu_h}{\Theta}}}{\lfloor\nu_1 \ldots \lfloor\nu_h} \int_{\text{phases}}^{\text{all}} \ldots \int e^{-\frac{\epsilon}{\Theta}} dp_1 \ldots dq_n, \qquad (505)$$

where the multiple sum indicated by $\Sigma_{\nu_1} \ldots \Sigma_{\nu_h}$ includes all terms obtained by giving to each of the symbols $\nu_1 \ldots \nu_h$ all integral values from zero upward, and the multiple integral (which is to be evaluated separately for each term of the multiple sum) is to be extended over all the (specific) phases of the system having the specified numbers of particles of the various kinds. The multiple integral in the last equation is what we have represented by $e^{-\frac{\psi}{\Theta}}$. See equation (92). We may therefore write

$$e^{-\frac{\Omega}{\Theta}} = \Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \frac{e^{\frac{\mu_1\nu_1 \ldots \mu_h\nu_h - \psi}{\Theta}}}{\lfloor\nu_1 \ldots \lfloor\nu_h}. \qquad (506)$$

It should be observed that the summation includes a term in which all the symbols $\nu_1 \ldots \nu_h$ have the value zero. We must therefore recognize in a certain sense a system consisting of no particles, which, although a barren subject of study in itself, cannot well be excluded as a particular case of a system of a variable number of particles. In this case ϵ is constant, and there are no integrations to be performed. We have therefore*

$$e^{-\frac{\psi}{\Theta}} = e^{-\frac{\epsilon}{\Theta}}, \quad i.e., \quad \psi = \epsilon.$$

* This conclusion may appear a little strained. The original definition of ψ may not be regarded as fairly applying to systems of no degrees of freedom. We may therefore prefer to regard these equations as defining ψ in this case.

The value of ϵ_p is of course zero in this case. But the value of ϵ_q contains an arbitrary constant, which is generally determined by considerations of convenience, so that ϵ_q and ϵ do not necessarily vanish with $\nu_1, \ldots \nu_h$.

Unless $-\Omega$ has a finite value, our formulae become illusory. We have already, in considering petit ensembles canonically distributed, found it necessary to exclude cases in which $-\psi$ has not a finite value.* The same exclusion would here make $-\psi$ finite for any finite values of $\nu_1 \ldots \nu_h$. This does not necessarily make a multiple series of the form (506) finite. We may observe, however, that if for all values of $\nu_1 \ldots \nu_h$

$$-\psi \leqq c_0 + c_1 \nu_1, \ldots + c_h \nu_h, \tag{507}$$

where $c_0, c_1, \ldots c_h$ are constants or functions of Θ,

$$e^{-\frac{\Omega}{\Theta}} \leqq \Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \frac{e^{\frac{c_0+(\mu_1+c_1)\nu_1 \ldots +(\mu_h+c_h)\nu_h}{\Theta}}}{\lfloor \nu_1 \ldots \lfloor \nu_h}$$

i. e.,
$$e^{-\frac{\Omega}{\Theta}} \leqq e^{\frac{c_0}{\Theta}} \Sigma_{\nu_1} \frac{e^{\frac{\mu_1+c_1}{\Theta}\nu_1}}{\lfloor \nu_1} \ldots \Sigma_{\nu_h} \frac{e^{\frac{\mu_h+c_h}{\Theta}\nu_h}}{\lfloor \nu_h}$$

i. e.,
$$e^{-\frac{\Omega}{\Theta}} \leqq e^{\frac{c_0}{\Theta}} e^{e^{\frac{\mu_1+c_1}{\Theta}}} \ldots e^{e^{\frac{\mu_h+\nu_h}{\Theta}}}$$

i. e.,
$$-\frac{\Omega}{\Theta} \leqq \frac{c_0}{\Theta} + e^{\frac{\mu_1+c_1}{\Theta}} \ldots + e^{\frac{\mu_h+c_h}{\Theta}} \tag{508}$$

The value of $-\Omega$ will therefore be finite, when the condition (507) is satisfied. If therefore we assume that $-\Omega$ is finite, we do not appear to exclude any cases which are analogous to those of nature.†

The interest of the ensemble which has been described lies in the fact that it may be in statistical equilbrium, both in

* See Chapter IV, page 35.

† If the external coördinates determine a certain volume within which the system is confined, the contrary of (507) would imply that we could obtain an infinite amount of work by crowding an infinite quantity of matter into a finite volume.

respect to exchange of energy and exchange of particles, with other grand ensembles canonically distributed and having the same values of Θ and of the coefficients μ_1, μ_2, etc., when the circumstances are such that exchange of energy and of particles are possible, and when equilibrium would not subsist, were it not for equal values of these constants in the two ensembles.

With respect to the exchange of energy, the case is exactly the same as that of the petit ensembles considered in Chapter IV, and needs no especial discussion. The question of exchange of particles is to a certain extent analogous, and may be treated in a somewhat similar manner. Let us suppose that we have two grand ensembles canonically distributed with respect to specific phases, with the same value of the modulus and of the coefficients $\mu_1 \ldots \mu_h$, and let us consider the ensemble of all the systems obtained by combining each system of the first ensemble with each of the second.

The probability-coefficient of a generic phase in the first ensemble may be expressed by

$$e^{\frac{\Omega' + \mu_1 \nu_1' \ldots + \mu_h \nu_h' - \epsilon'}{\Theta}} \tag{509}$$

The probability-coefficient of a specific phase will then be expressed by

$$\frac{e^{\frac{\Omega' + \mu_1 \nu_1' \ldots + \mu_h \nu_h' - \epsilon'}{\Theta}}}{\lfloor \nu_1' \ldots \lfloor \nu_h'}, \tag{510}$$

since each generic phase comprises $\lfloor \nu_1 \ldots \lfloor \nu_h$ specific phases. In the second ensemble the probability-coefficients of the generic and specific phases will be

$$e^{\frac{\Omega'' + \mu_1 \nu_1'' \ldots + \mu_h \nu_h'' - \epsilon''}{\Theta}}, \tag{511}$$

and

$$\frac{e^{\frac{\Omega'' + \mu_1 \nu_1'' \ldots + \mu_h \nu_h'' - \epsilon''}{\Theta}}}{\lfloor \nu_1'' \ldots \lfloor \nu_h''}. \tag{512}$$

The probability-coefficient of a generic phase in the third ensemble, which consists of systems obtained by regarding each system of the first ensemble combined with each of the second as forming a system, will be the product of the probability-coefficients of the generic phases of the systems combined, and will therefore be represented by the formula

$$e^{\frac{\Omega''' + \mu_1 \nu_1''' \ldots + \mu_h \nu_h''' - \epsilon'''}{\Theta}} \tag{513}$$

where $\Omega''' = \Omega' + \Omega''$, $\epsilon''' = \epsilon' + \epsilon''$, $\nu_1''' = \nu_1' + \nu_1''$, etc. It will be observed that ν_1''', etc., represent the numbers of particles of the various kinds in the third ensemble, and ϵ''' its energy; also that Ω''' is a constant. The third ensemble is therefore canonically distributed with respect to generic phases.

If all the systems in the same generic phase in the third ensemble were equably distributed among the $\lfloor\underline{\nu_1'''} \ldots \lfloor\underline{\nu_h'''}$ specific phases which are comprised in the generic phase, the probability-coefficient of a specific phase would be

$$\frac{e^{\frac{\Omega''' + \mu_1 \nu_1''' \ldots + \mu_h \nu_h''' - \epsilon'''}{\Theta}}}{\lfloor\underline{\nu_1'''} \ldots \lfloor\underline{\nu_h'''}}. \tag{514}$$

In fact, however, the probability-coefficient of any specific phase which occurs in the third ensemble is

$$\frac{e^{\frac{\Omega''' + \mu_1 \nu_1''' \ldots + \mu_h \nu_h''' - \epsilon'''}{\Theta}}}{\lfloor\underline{\nu_1'} \ldots \lfloor\underline{\nu_h'} \lfloor\underline{\nu_1''} \ldots \lfloor\underline{\nu_h''}}, \tag{515}$$

which we get by multiplying the probability-coefficients of specific phases in the first and second ensembles. The difference between the formulae (514) and (515) is due to the fact that the generic phases to which (513) relates include not only the specific phases occurring in the third ensemble and having the probability-coefficient (515), but also all the specific phases obtained from these by interchange of similar particles between two combined systems. Of these the proba-

bility-coefficient is evidently zero, as they do not occur in the ensemble.

Now this third ensemble is in statistical equilibrium, with respect both to specific and generic phases, since the ensembles from which it is formed are so. This statistical equilibrium is not dependent on the equality of the modulus and the co-efficients $\mu_1, \ldots \mu_h$ in the first and second ensembles. It depends only on the fact that the two original ensembles were separately in statistical equilibrium, and that there is no interaction between them, the combining of the two ensembles to form a third being purely nominal, and involving no physical connection. This independence of the systems, determined physically by forces which prevent particles from passing from one system to the other, or coming within range of each other's action, is represented mathematically by infinite values of the energy for particles in a space dividing the systems. Such a space may be called a diaphragm.

If we now suppose that, when we combine the systems of the two original ensembles, the forces are so modified that the energy is no longer infinite for particles in all the space forming the diaphragm, but is diminished in a part of this space, so that it is possible for particles to pass from one system to the other, this will involve a change in the function ϵ''' which represents the energy of the combined systems, and the equation $\epsilon''' = \epsilon' + \epsilon''$ will no longer hold. Now if the coefficient of probability in the third ensemble were represented by (513) with this new function ϵ''', we should have statistical equilibrium, with respect to generic phases, although not to specific. But this need involve only a trifling change in the distribution of the third ensemble,* a change represented by the addition of comparatively few systems in which the transference of particles is taking place to the immense number

* It will be observed that, so far as the distribution is concerned, very large and infinite values of ϵ (for certain phases) amount to nearly the same thing, — one representing the total and the other the nearly total exclusion of the phases in question. An infinite change, therefore, in the value of ϵ (for certain phases) may represent a vanishing change in the distribution.

obtained by combining the two original ensembles. The difference between the ensemble which would be in statistical equilibrium, and that obtained by combining the two original ensembles may be diminished without limit, while it is still possible for particles to pass from one system to another. In this sense we may say that the ensemble formed by combining the two given ensembles may still be regarded as in a state of (approximate) statistical equilibrium with respect to generic phases, when it has been made possible for particles to pass between the systems combined, and when statistical equilibrium for specific phases has therefore entirely ceased to exist, and when the equilibrium for generic phases would also have entirely ceased to exist, if the given ensembles had not been canonically distributed, with respect to generic phases, with the same values of Θ and $\mu_1, \ldots \mu_h$.

It is evident also that considerations of this kind will apply separately to the several kinds of particles. We may diminish the energy in the space forming the diaphragm for one kind of particle and not for another. This is the mathematical expression for a "semipermeable" diaphragm. The condition necessary for statistical equilibrium where the diaphragm is permeable only to particles to which the suffix $(\ )_1$ relates will be fulfilled when μ_1 and Θ have the same values in the two ensembles, although the other coefficients μ_2, etc., may be different.

This important property of grand ensembles with canonical distribution will supply the motive for a more particular examination of the nature of such ensembles, and especially of the comparative numbers of systems in the several petit ensembles which make up a grand ensemble, and of the average values in the grand ensemble of some of the most important quantities, and of the average squares of the deviations from these average values.

The probability that a system taken at random from a grand ensemble canonically distributed will have exactly $\nu_1, \ldots \nu_h$ particles of the various kinds is expressed by the multiple integral

$$\int_{\text{phases}}^{\text{all}} \cdots \int \frac{e^{\frac{\Omega+\mu_1\nu_1\cdots+\mu_h\nu_h-\epsilon}{\Theta}}}{\lfloor\nu_1 \cdots \lfloor\nu_h} \, dp_1 \cdots dq_n \tag{516}$$

or

$$\frac{e^{\frac{\Omega+\mu_1\nu_1\cdots+\mu_h\nu_h-\psi}{\Theta}}}{\lfloor\nu_1 \cdots \lfloor\nu_h}. \tag{517}$$

This may be called the probability of the petit ensemble $(\nu_1, \ldots \nu_h)$. The sum of all such probabilities is evidently unity. That is,

$$\Sigma_{\nu_1} \cdots \Sigma_{\nu_h} \frac{e^{\frac{\Omega+\mu_1\nu_1\cdots+\mu_h\nu_h-\psi}{\Theta}}}{\lfloor\nu_1 \cdots \lfloor\nu_h} = 1, \tag{518}$$

which agrees with (506).

The average value in the grand ensemble of any quantity u, is given by the formula

$$\bar{u} = \Sigma_{\nu_1} \cdots \Sigma_{\nu_h} \int_{\text{phases}}^{\text{all}} \cdots \int \frac{u\, e^{\frac{\Omega+\mu_1\nu_1\cdots+\mu_h\nu_h-\epsilon}{\Theta}}}{\lfloor\nu_1 \cdots \lfloor\nu_h} \, dp_1 \cdots dq_n. \tag{519}$$

If u is a function of $\nu_1, \ldots \nu_h$ alone, *i. e.*, if it has the same value in all systems of any same petit ensemble, the formula reduces to

$$\bar{u} = \Sigma_{\nu_1} \cdots \Sigma_{\nu_h} \frac{u\, e^{\frac{\Omega+\mu_1\nu_1\cdots+\mu_h\nu_h-\psi}{\Theta}}}{\lfloor\nu_1 \cdots \lfloor\nu_h}. \tag{520}$$

Again, if we write $\bar{u}|_{\text{grand}}$ and $\bar{u}|_{\text{petit}}$ to distinguish averages in the grand and petit ensembles, we shall have

$$\bar{u}|_{\text{grand}} = \Sigma_{\nu_1} \cdots \Sigma_{\nu_h} \bar{u}|_{\text{petit}} \frac{e^{\frac{\Omega+\mu_1\nu_1\cdots+\mu_h\nu_h-\psi}{\Theta}}}{\lfloor\nu_1 \cdots \lfloor\nu_h}. \tag{521}$$

In this chapter, in which we are treating of grand ensembles, $\bar{u}$ will always denote the average for a grand ensemble. In the preceding chapters, $\bar{u}$ has always denoted the average for a petit ensemble.

Equation (505), which we repeat in a slightly different form, viz.,

$$e^{-\frac{\Omega}{\Theta}} = \Sigma_{\nu_1} \dots \Sigma_{\nu_h} \int_{\text{phases}}^{\text{all}} \dots \int \frac{e^{\frac{\mu_1\nu_1 \dots + \mu_h\nu_h - \epsilon}{\Theta}}}{\lfloor\nu_1 \dots \lfloor\nu_h} dp_1 \dots dq_n, \quad (522)$$

shows that Ω is a function of Θ and $\mu_1, \dots \mu_h$; also of the external coördinates a_1, a_2, etc., which are involved implicitly in ϵ. If we differentiate the equation regarding all these quantities as variable, we have

$$\begin{aligned} e^{-\frac{\Omega}{\Theta}}\left(-\frac{d\Omega}{\Theta} + \frac{\Omega}{\Theta^2} d\Theta\right) = \\ -\frac{d\Theta}{\Theta^2} \Sigma_{\nu_1} \dots \Sigma_{\nu_h} \int_{\text{phases}}^{\text{all}} \dots \int \frac{(\mu_1\nu_1 \dots + \mu_h\nu_h - \epsilon)\, e^{\frac{\mu_1\nu_1 \dots + \mu_h\nu_h - \epsilon}{\Theta}}}{\lfloor\nu_1 \dots \lfloor\nu_h} dp_1 \dots dq_n \\ + \frac{d\mu_1}{\Theta} \Sigma_{\nu_1} \dots \Sigma_{\nu_h} \int_{\text{phases}}^{\text{all}} \dots \int \frac{\nu_1 e^{\frac{\mu_1\nu_1 \dots + \mu_h\nu_h - \epsilon}{\Theta}}}{\lfloor\nu_1 \dots \lfloor\nu_h} dp_1 \dots dq_n \\ + \text{etc.} \\ -\frac{da_1}{\Theta} \Sigma_{\nu_1} \dots \Sigma_{\nu_h} \int_{\text{phases}}^{\text{all}} \dots \int \frac{d\epsilon}{da_1} \frac{e^{\frac{\mu_1\nu_1 \dots + \mu_h\nu_h - \epsilon}{\Theta}}}{\lfloor\nu_1 \dots \lfloor\nu_h} dp_1 \dots dq_n, \\ - \text{etc.} \end{aligned} \quad (523)$$

If we multiply this equation by $e^{\frac{\Omega}{\Theta}}$, and set as usual A_1, A_2, etc., for $-d\epsilon/da_1$, $-d\epsilon/da_2$, etc., we get in virtue of the law expressed by equation (519),

$$\begin{aligned} -\frac{d\Omega}{\Theta} + \frac{\Omega}{\Theta^2} d\Theta = -\frac{d\Theta}{\Theta^2}(\mu_1 \bar{\nu}_1 \dots + \mu_h \bar{\nu}_h - \bar{\epsilon}) \\ + \frac{d\mu_1}{\Theta} \bar{\nu}_1 + \frac{d\mu_2}{\Theta} \bar{\nu}_2 + \text{etc.} \\ + \frac{da_1}{\Theta} \bar{A}_1 + \frac{da_2}{\Theta} \bar{A}_2 + \text{etc.}; \end{aligned} \quad (524)$$

that is,

$$d\Omega = \frac{\Omega + \mu_1 \bar{\nu}_1 \ldots \mu_h \bar{\nu}_h - \bar{\epsilon}}{\Theta} d\Theta - \Sigma \bar{\nu}_1 \, d\mu_1 - \Sigma \bar{A}_1 \, da_1. \quad (525)$$

Since equation (503) gives

$$\frac{\Omega + \mu_1 \bar{\nu}_1 \ldots + \mu_h \bar{\nu}_h - \bar{\epsilon}}{\Theta} = \bar{\mathrm{H}}, \quad (526)$$

the preceding equation may be written

$$d\Omega = \bar{\mathrm{H}} \, d\Theta - \Sigma \bar{\nu}_1 \, d\mu_1 - \Sigma \bar{A}_1 \, da_1. \quad (527)$$

Again, equation (526) gives

$$d\Omega + \Sigma \mu_1 \, d\bar{\nu}_1 + \Sigma \bar{\nu}_1 \, d\mu_1 - d\bar{\epsilon} = \Theta \, d\bar{\mathrm{H}} + \bar{\mathrm{H}} \, d\Theta. \quad (528)$$

Eliminating $d\Omega$ from these equations, we get

$$d\bar{\epsilon} = -\Theta \, d\bar{\mathrm{H}} + \Sigma \mu_1 \, d\bar{\nu}_1 - \Sigma \bar{A}_1 \, da_1. \quad (529)$$

If we set $$\Psi = \bar{\epsilon} + \Theta \bar{\mathrm{H}}, \quad (530)$$

$$d\Psi = d\epsilon + \Theta \, d\bar{\mathrm{H}} + \bar{\mathrm{H}} \, d\Theta, \quad (531)$$

we have $$d\Psi = \bar{\mathrm{H}} \, d\Theta + \Sigma \mu_1 \, d\bar{\nu}_1 - \Sigma \bar{A}_1 \, da_1. \quad (532)$$

The corresponding thermodynamic equations are

$$d\epsilon = T \, d\eta + \Sigma \mu_1 \, dm_1 - \Sigma A_1 \, da_1, \quad (533)$$

$$\psi = \epsilon - T\eta, \quad (534)$$

$$d\psi = -\eta \, dT + \Sigma \mu_1 \, dm_1 - \Sigma A_1 \, da_1. \quad (535)$$

These are derived from the thermodynamic equations (114) and (117) by the addition of the terms necessary to take account of variation in the quantities (m_1, m_2, etc.) of the several substances of which a body is composed. The correspondence of the equations is most perfect when the component substances are measured in such units that m_1, m_2, etc., are proportional to the numbers of the different kinds of molecules or atoms. The quantities μ_1, μ_2, etc., in these thermodynamic equations may be defined as differential coefficients by either of the equations in which they occur.*

* Compare Transactions Connecticut Academy, Vol. III, pages 116 ff.

If we compare the statistical equations (529) and (532) with (114) and (112), which are given in Chapter IV, and discussed in Chapter XIV, as analogues of thermodynamic equations, we find considerable difference. Beside the terms corresponding to the additional terms in the thermodynamic equations of this chapter, and beside the fact that the averages are taken in a grand ensemble in one case and in a petit in the other, the analogues of entropy, H and η, are quite different in definition and value. We shall return to this point after we have determined the order of magnitude of the usual anomalies of $\nu_1, \ldots \nu_h$.

If we differentiate equation (518) with respect to μ_1, and multiply by Θ, we get

$$\Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \left(\frac{d\Omega}{d\mu_1} + \nu_1 \right) \frac{e^{\frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \psi}{\Theta}}}{\lfloor \nu_1 \ldots \lfloor \nu_h} = 0, \qquad (536)$$

whence $d\Omega/d\mu_1 = -\bar{\nu}_1$, which agrees with (527). Differentiating again with respect to μ_1, and to μ_2, and setting

$$\frac{d\Omega}{d\mu_1} = -\bar{\nu}_1, \quad \frac{d\Omega}{d\mu_2} = -\bar{\nu}_2,$$

we get

$$\Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \left(\frac{d^2\Omega}{d\mu_1^2} + \frac{(\nu_1 - \bar{\nu}_1)^2}{\Theta} \right) \frac{e^{\frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \psi}{\Theta}}}{\lfloor \nu_1 \ldots \lfloor \nu_h} = 0, \qquad (537)$$

$$\Sigma_{\nu_1} \ldots \Sigma_{\nu_h} \left(\frac{d^2\Omega}{d\mu_1 d\mu_2} + \frac{(\nu_1 - \bar{\nu}_1)(\nu_2 - \bar{\nu}_2)}{\Theta} \right) \frac{e^{\frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \psi}{\Theta}}}{\lfloor \nu_1 \ldots \lfloor \nu_h} = 0. \qquad (538)$$

The first members of these equations represent the average values of the quantities in the principal parentheses. We have therefore

$$\overline{(\nu_1 - \bar{\nu}_1)^2} = \overline{\nu_1^2} - \bar{\nu}_1^2 = -\Theta \frac{d^2\Omega}{d\mu_1^2} = \Theta \frac{d\bar{\nu}_1}{d\mu_1}, \qquad (539)$$

$$\overline{(\nu_1 - \bar{\nu}_1)(\nu_2 - \bar{\nu}_2)} = \overline{\nu_1 \nu_2} - \bar{\nu}_1 \bar{\nu}_2 = -\Theta \frac{d^2\Omega}{d\mu_1 d\mu_2} = \Theta \frac{d\bar{\nu}_1}{d\mu_2} = \Theta \frac{d\bar{\nu}_2}{d\mu_1}. \qquad (540)$$

From equation (539) we may get an idea of the order of magnitude of the divergences of ν_1 from its average value in the ensemble, when that average value is great. The equation may be written

$$\frac{(\nu_1 - \bar{\nu}_1)^2}{\bar{\nu}_1^2} = \frac{\Theta}{\bar{\nu}_1^2} \frac{d\bar{\nu}_1}{d\mu_1}, \tag{541}$$

The second member of this equation will in general be small when $\bar{\nu}_1$ is great. Large values are not necessarily excluded, but they must be confined within very small limits with respect to μ. For if

$$\frac{\overline{(\nu_1 - \bar{\nu}_1)^2}}{\bar{\nu}_1^2} > \frac{1}{\bar{\nu}_1^{\frac{1}{2}}}, \tag{542}$$

for all values of μ_1 between the limits μ_1' and μ_1'', we shall have between the same limits

$$\frac{\Theta}{\bar{\nu}_1^{\frac{3}{2}}} d\bar{\nu}_1 > d\mu_1, \tag{543}$$

and therefore

$$\tfrac{1}{2}\Theta\left(\frac{1}{\bar{\nu}_1'^{\frac{1}{2}}} - \frac{1}{\bar{\nu}_1''^{\frac{1}{2}}}\right) > \mu_1'' - \mu_1'. \tag{544}$$

The difference $\mu_1'' - \mu_1'$ is therefore numerically a very small quantity. To form an idea of the importance of such a difference, we should observe that in formula (498) μ_1 is multiplied by ν_1 and the product subtracted from the energy. A very small difference in the value of μ_1 may therefore be important. But since $\nu\Theta$ is always less than the kinetic energy of the system, our formula shows that $\mu_1'' - \mu_1'$, even when multiplied by $\bar{\nu}_1'$ or $\bar{\nu}_1''$, may still be regarded as an insensible quantity.

We can now perceive the leading characteristics with respect to properties sensible to human faculties of such an ensemble as we are considering (a grand ensemble canonically distributed), when the average numbers of particles of the various kinds are of the same order of magnitude as the number of molecules in the bodies which are the subject of physical

experiment. Although the ensemble contains systems having the widest possible variations in respect to the numbers of the particles which they contain, these variations are practically contained within such narrow limits as to be insensible, except for particular values of the constants of the ensemble. This exception corresponds precisely to the case of nature, when certain thermodynamic quantities corresponding to Θ, μ_1, μ_2, etc., which in general determine the separate densities of various components of a body, have certain values which make these densities indeterminate, in other words, when the conditions are such as determine coexistent phases of matter. Except in the case of these particular values, the grand ensemble would not differ to human faculties of perception from a petit ensemble, viz., any one of the petit ensembles which it contains in which $\bar{\nu}_1$, $\bar{\nu}_2$, etc., do not sensibly differ from their average values.

Let us now compare the quantities H and η, the average values of which (in a grand and a petit ensemble respectively) we have seen to correspond to entropy. Since

$$\mathrm{H} = \frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \epsilon}{\Theta},$$

and

$$\eta = \frac{\psi - \epsilon}{\Theta},$$

$$\mathrm{H} - \eta = \frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \psi}{\Theta}. \qquad (545)$$

A part of this difference is due to the fact that H relates to generic phases and η to specific. If we write η_{gen} for the index of probability for generic phases in a petit ensemble, we have

$$\eta_{\text{gen}} = \eta + \log \lfloor \nu_1 \ldots \lfloor \nu_h, \qquad (546)$$

$$\mathrm{H} - \eta = \mathrm{H} - \eta_{\text{gen}} + \log \lfloor \nu_1 \ldots \lfloor \nu_h, \qquad (547)$$

$$\mathrm{H} - \eta_{\text{gen}} = \frac{\Omega + \mu_1 \nu_1 \ldots + \mu_h \nu_h - \psi}{\Theta} - \log \lfloor \nu_1 \ldots \lfloor \nu_h. \qquad (548)$$

This is the logarithm of the probability of the petit ensemble $(\nu_1 \ldots \nu_h)$.* If we set

* See formula (517).

$$\frac{\psi_{\text{gen}} - \epsilon}{\Theta} = \eta_{\text{gen}}, \tag{549}$$

which corresponds to the equation

$$\frac{\psi - \epsilon}{\Theta} = \eta, \tag{550}$$

we have $\psi_{\text{gen}} = \psi + \Theta \log \lfloor\underline{\nu_1} \ldots \lfloor\underline{\nu_h}$,

and $$\mathrm{H} - \eta_{\text{gen}} = \frac{\Omega + \mu_1\nu_1 \ldots + \mu_h\nu_h - \psi_{\text{gen}}}{\Theta}. \tag{551}$$

This will have a maximum when *

$$\frac{d\psi_{\text{gen}}}{d\nu_1} = \mu_1, \qquad \frac{d\psi_{\text{gen}}}{d\nu_2} = \mu_2, \qquad \text{etc.} \tag{552}$$

Distinguishing values corresponding to this maximum by accents, we have approximately, when $\nu_1, \ldots \nu_h$ are of the same order of magnitude as the numbers of molecules in ordinary bodies,

$$\begin{aligned}\mathrm{H} - \eta_{\text{gen}} &= \frac{\Omega + \mu_1\nu_1 \ldots + \mu_h\nu_h - \psi_{\text{gen}}}{\Theta} \\ &= \frac{\Omega + \mu_1\nu_1' \ldots + \mu_h\nu_h' - \psi_{\text{gen}}'}{\Theta} \\ &\quad - \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1^2}\right)' \frac{(\Delta\nu_1)^2}{2\Theta} - \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1 d\nu_2}\right)' \frac{\Delta\nu_1\Delta\nu_2}{\Theta} \ldots - \left(\frac{d^2\psi_{\text{gen}}}{d\nu_h^2}\right)' \frac{(\Delta\nu_h)^2}{2\Theta},\end{aligned} \tag{553}$$

$$e^{\mathrm{H}-\eta_{\text{gen}}} = e^{C} e^{-\left(\frac{d^2\psi_{\text{gen}}}{d\nu_1^2}\right)' \frac{(\Delta\nu_1)^2}{2\Theta} - \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1 d\nu_2}\right)' \frac{\Delta\nu_1\Delta\nu_2}{\Theta} \ldots - \left(\frac{d^2\psi}{d\nu_h^2}\right)' \frac{(\Delta\nu_h)^2}{2\Theta}}, \tag{554}$$

where $$C = \frac{\Omega + \mu_1\nu_1' \ldots + \mu_h\nu_h' - \psi_{\text{gen}}'}{\Theta}, \tag{555}$$

and $$\Delta\nu_1 = \nu_1 - \nu_1', \qquad \Delta\nu_2 = \nu_2 - \nu_2', \qquad \text{etc.} \tag{556}$$

This is the probability of the system $(\nu_1 \ldots \nu_h)$. The probabilty that the values of $\nu_1, \ldots \nu_h$ lie within given limits is given by the multiple integral

* Strictly speaking, ψ_{gen} is not determined as function of $\nu_1, \ldots \nu_h$, except for integral values of these variables. Yet we may suppose it to be determined as a continuous function by any suitable process of interpolation.

$$\int \cdots \int e^{C} e^{-\left(\frac{d^2\psi_{\text{gen}}}{d\nu_1^2}\right)'\frac{(\Delta\nu_1)^2}{2\Theta} - \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1 d\nu_2}\right)'\frac{\Delta\nu_1\Delta\nu_2}{\Theta} \cdots - \left(\frac{d^2\psi_{\text{gen}}}{d\nu_h^2}\right)'\frac{(\Delta\nu_h)^2}{2\Theta}} d\nu_1 \ldots d\nu_h. \tag{557}$$

This shows that the distribution of the grand ensemble with respect to the values of $\nu_1, \ldots \nu_h$ follows the "law of errors" when $\nu_1', \ldots \nu_h'$ are very great. The value of this integral for the limits $\pm\infty$ should be unity. This gives

$$\frac{e^{C}(2\pi\Theta)^{\frac{h}{2}}}{D^{\frac{1}{2}}} = 1, \tag{558}$$

or

$$C = \tfrac{1}{2}\log D - \frac{h}{2}\log(2\pi\Theta), \tag{559}$$

where

$$D = \begin{vmatrix} \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1^2}\right)' & \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1 d\nu_2}\right)' & \cdots & \left(\frac{d^2\psi_{\text{gen}}}{d\nu_1 d\nu_h}\right)' \\ \left(\frac{d^2\psi_{\text{gen}}}{d\nu_2 d\nu_1}\right)' & \left(\frac{d^2\psi_{\text{gen}}}{d\nu_2^2}\right)' & \cdots & \left(\frac{d^2\psi_{\text{gen}}}{d\nu_2 d\nu_h}\right)' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{d^2\psi_{\text{gen}}}{d\nu_h d\nu_1}\right)' & \left(\frac{d^2\psi_{\text{gen}}}{d\nu_h d\nu_2}\right)' & \cdots & \left(\frac{d^2\psi_{\text{gen}}}{d\nu_h^2}\right)' \end{vmatrix} \tag{560}$$

that is,

$$D = \begin{vmatrix} \left(\frac{d\mu_1}{d\nu_1}\right)' & \left(\frac{d\mu_1}{d\nu_2}\right)' & \cdots & \left(\frac{d\mu_1}{d\nu_h}\right)' \\ \left(\frac{d\mu_2}{d\nu_1}\right)' & \left(\frac{d\mu_2}{d\nu_2}\right)' & \cdots & \left(\frac{d\mu_2}{d\nu_h}\right)' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \left(\frac{d\mu_h}{d\nu_1}\right)' & \left(\frac{d\mu_h}{d\nu_2}\right)' & \cdots & \left(\frac{d\mu_h}{d\nu_h}\right)' \end{vmatrix} \tag{561}$$

Now, by (553), we have for the first approximation

$$\bar{\mathrm{H}} - \bar{\eta}_{\text{gen}} = C = \tfrac{1}{2}\log D - \frac{h}{2}\log(2\pi\Theta), \tag{562}$$

and if we divide by the constant K,* to reduce these quantities to the usual unit of entropy,

$$\frac{\bar{\mathrm{H}} - \bar{\eta}_{\text{gen}}}{K} = \frac{\log D - h\log(2\pi\Theta)}{2K} \tag{563}$$

* See page 184–186.

This is evidently a negligible quantity, since K is of the same order of magnitude as the number of molecules in ordinary bodies. It is to be observed that $\bar{\eta}_{\text{gen}}$ is here the average in the grand ensemble, whereas the quantity which we wish to compare with H is the average in a petit ensemble. But as we have seen that in the case considered the grand ensemble would appear to human observation as a petit ensemble, this distinction may be neglected.

The differences therefore, in the case considered, between the quantities which may be represented by the notations *

$$\overline{H_{\text{gen}}}\Big|_{\text{grand}}, \quad \overline{\eta_{\text{gen}}}\Big|_{\text{grand}}, \quad \overline{\eta_{\text{gen}}}\Big|_{\text{petit}}$$

are not sensible to human faculties. The difference

$$\overline{\eta_{\text{gen}}}\Big|_{\text{petit}} - \overline{\eta_{\text{spec}}}\Big|_{\text{petit}} = \lfloor \nu_1 \ldots \lfloor \nu_h,$$

and is therefore constant, so long as the numbers $\nu_1, \ldots \nu_h$ are constant. For constant values of these numbers, therefore, it is immaterial whether we use the average of η_{gen} or of η for entropy, since this only affects the arbitrary constant of integration which is added to entropy. But when the numbers $\nu_1, \ldots \nu_h$ are varied, it is no longer possible to use the index for specific phases. For the principle that the entropy of any body has an arbitrary additive constant is subject to limitation, when different quantities of the same substance are concerned. In this case, the constant being determined for one quantity of a substance, is thereby determined for all quantities of the same substance.

To fix our ideas, let us suppose that we have two identical fluid masses in contiguous chambers. The entropy of the whole is equal to the sum of the entropies of the parts, and double that of one part. Suppose a valve is now opened, making a communication between the chambers. We do not regard this as making any change in the entropy, although the masses of gas or liquid diffuse into one another, and although the same process of diffusion would increase the

* In this paragraph, for greater distinctness, $\overline{H_{\text{gen}}}\Big|_{\text{grand}}$ and $\overline{\eta_{\text{spec}}}\Big|_{\text{petit}}$ have been written for the quantities which elsewhere are denoted by $\bar{H}$ and $\bar{\eta}$.

entropy, if the masses of fluid were different. It is evident, therefore, that it is equilibrium with respect to generic phases, and not with respect to specific, with which we have to do in the evaluation of entropy, and therefore, that we must use the average of H or of η_{gen}, and not that of η, as the equivalent of entropy, except in the thermodynamics of bodies in which the number of molecules of the various kinds is constant.

PART TWO

DYNAMICS
VECTOR ANALYSIS AND MULTIPLE ALGEBRA
ELECTROMAGNETIC THEORY OF LIGHT
ETC.

PREFATORY NOTE TO PART TWO

In a few cases slight corrections had been made by the author in his own copies of the papers. These changes, together with the correction of obvious misprints in the originals, have been incorporated in the present edition without comment. Where for the sake of clearness it has seemed desirable to insert a word or two in a footnote or in the text itself, the addition has been indicated by enclosing it within square brackets [ ], a sign which is otherwise used only in the formulæ.

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I.

ON THE FUNDAMENTAL FORMULÆ OF DYNAMICS.

[*American Journal of Mathematics*, vol. II. pp. 49-64, 1879.]

Formation of a new Indeterminate Formula of Motion by the Substitution of the Variations of the Components of Acceleration for the Variations of the Coordinates in the usual Formula.

The laws of motion are frequently expressed by an equation of the form

$$\Sigma[(X-m\ddot{x})\delta x+(Y-m\ddot{y})\delta y+(Z-m\ddot{z})\delta z]=0, \qquad (1)$$

in which

m denotes the mass of a particle of the system considered,

x, y, z its rectangular coordinates,

$\ddot{x}$, $\ddot{y}$, $\ddot{z}$ the second differential coefficients of the coordinates with respect to the time,

X, Y, Z the components of the forces acting on the particle,

δx, δy, δz any arbitrary variations of the coordinates which are simultaneously possible, and

Σ a summation with respect to all the particles of the system.

It is evident that we may substitute for δx, δy, δz any other expressions which are capable of the same and only of the same sets of simultaneous values.

Now if the nature of the system is such that certain functions A, B, etc. of the coordinates must be constant, or given functions of the time, we have

$$\left.\begin{aligned}&\Sigma\left(\frac{dA}{dx}\delta x+\frac{dA}{dy}\delta y+\frac{dA}{dz}\delta z\right)=0,\\&\Sigma\left(\frac{dB}{dx}\delta x+\frac{dB}{dy}\delta y+\frac{dB}{dz}\delta z\right)=0,\\&\text{etc.}\end{aligned}\right\} \qquad (2)$$

These are the *equations of condition*, to which the variations in the general equation of motion (1) are subject. But if A is constant or a determined function of the time, the same must be true of $\dot{A}$ and $\ddot{A}$. Now

$$\dot{A}=\Sigma\left(\frac{dA}{dx}\dot{x}+\frac{dA}{dy}\dot{y}+\frac{dA}{dz}\dot{z}\right)$$

and

$$\ddot{A}=\Sigma\left(\frac{dA}{dx}\ddot{x}+\frac{dA}{dy}\ddot{y}+\frac{dA}{dz}\ddot{z}\right)+H,$$

where H represents terms containing only the second differential coefficients of A with respect to the coordinates, and the first differential coefficients of the coordinates with respect to the time. Therefore, if we conceive of a variation affecting the accelerations of the particles at the time considered, but not their positions or velocities, we have

$$\left.\begin{aligned} \delta\ddot{A} &= \Sigma\left(\frac{dA}{dx}\delta\ddot{x}+\frac{dA}{dy}\delta\ddot{y}+\frac{dA}{dz}\delta\ddot{z}\right)=0, \\ \text{and, in like manner,} \\ \delta\ddot{B} &= \Sigma\left(\frac{dB}{dx}\delta\ddot{x}+\frac{dB}{dy}\delta\ddot{y}+\frac{dB}{dz}\delta\ddot{z}\right)=0, \\ &\text{etc.} \end{aligned}\right\} \tag{3}$$

Comparing these equations with (2), we see that when the *accelerations* of the particles are regarded as subject to the variation denoted by δ, but not their positions or velocities, the possible values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ are subject to precisely the same restrictions as the values of δx, δy, δz, when the *positions* of the particles are regarded as variable. We may, therefore, write for the general equation of motion

$$\Sigma[(X-m\ddot{x})\delta\ddot{x}+(Y-m\ddot{y})\delta\ddot{y}+(Z-m\ddot{z})\delta\ddot{z}]=0, \tag{4}$$

regarding the positions and velocities of the particles as unaffected by the variation denoted by δ,—a condition which may be expressed by the equations

$$\left.\begin{aligned} \delta x=0, \quad \delta y=0, \quad \delta z=0, \\ \delta\dot{x}=0, \quad \delta\dot{y}=0, \quad \delta\dot{z}=0. \end{aligned}\right\} \tag{5}$$

We have so far supposed that the conditions which restrict the possible motions of the systems may be expressed by *equations* between the coordinates alone or the coordinates and the time. To extend the formula of motion to cases in which the conditions are expressed by the characters $\leqq$ or $\geqq$, we may write

$$\Sigma[(X-m\ddot{x})\delta\ddot{x}+(Y-m\ddot{y})\delta\ddot{y}+(Z-m\ddot{z})\delta\ddot{z}]\leqq 0, \tag{6}$$

The conditions which determine the possible values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ will not, in such cases, be entirely similar to those which determine the possible values of δx, δy, δz, when the coordinates are regarded as variable. Nevertheless, the laws of motion are correctly expressed by the formula (6), while the formula

$$\Sigma[(X-m\ddot{x})\delta x+(Y-m\ddot{y})\delta y+(Z-m\ddot{z})\delta z]\leqq 0, \tag{7}$$

does not, as naturally interpreted, give so complete and accurate an expression of the laws of motion.

This may be illustrated by a simple example.

Let it be required to find the acceleration of a material point, which, at a given instant, is moving with given velocity on the

frictionless surface of a body (which it cannot penetrate, but which it may leave), and is acted on by given forces. For simplicity, we may suppose that the normal to the surface, drawn outward from the moving point at the moment considered, is parallel to the axis of X and in the positive direction. The only restriction on the values of δx, δy, δz is that

$$\delta x \geqq 0.$$

Formula (7) will therefore give

$$\ddot{x} \geqq \frac{X}{m}, \quad \ddot{y} = \frac{Y}{m}, \quad \ddot{z} = \frac{Z}{m}.$$

The condition that the point shall not penetrate the body gives another condition for the value of $\ddot{x}$. If the point remains upon the surface, $\ddot{x}$ must have a certain value N, determined by the form of the surface and the velocity of the point. If the value of $\ddot{x}$ is less than this, the point must penetrate the body. Therefore,

$$\ddot{x} \geqq N.$$

But this does not suffice to determine the acceleration of the point.

Let us now apply formula (6) to the same problem. Since $\ddot{x}$ cannot be less than N,

$$\text{if } \ddot{x} = N, \quad \delta\ddot{x} \geqq 0.$$

This is the only restriction on the value of $\delta\ddot{x}$, for if $\ddot{x} > N$, the value of $\delta\ddot{x}$ is entirely arbitrary. Formula (6), therefore, requires that

$$\text{if } \ddot{x} = N, \quad \ddot{x} \geqq \frac{X}{m};$$

$$\text{but if } \ddot{x} > N, \quad \ddot{x} = \frac{X}{m}:$$

—that is (since $\ddot{x}$ cannot be less than N), that $\ddot{x}$ shall be equal to the greater of the quantities N and $\frac{X}{m}$, or to both, if they are equal,—and that

$$\ddot{y} = \frac{Y}{m}, \quad \ddot{z} = \frac{Z}{m}.$$

The values of $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are therefore entirely determined by this formula in connection with the conditions afforded by the constraints of the system.*

The following considerations will show that what is true in this case is also true in general, when the conditions to which the system

* The failure of the formula (7) in this case is rather apparent than real; for, although the formula apparently allows to $\ddot{x}$, at the instant considered, a value exceeding both N and $\frac{X}{m}$, it does not allow this for any interval, however short. For if $\ddot{x} < N$, the point will immediately leave the surface, and then the formula requires that $\ddot{x} = \frac{X}{m}$.

is subject are such that certain functions of the coordinates cannot exceed certain limits, either constant or variable with the time. If certain values of $\delta\ddot{x}, \delta\ddot{y}, \delta\ddot{z}$ (with unvaried values of x, y, z, and $\dot{x}, \dot{y}, \dot{z}$) are simultaneously possible at a given instant, equal or proportional values with the same signs must be possible for $\delta x, \delta y, \delta z$ immediately after the instant considered, and must satisfy formula (1), and therefore (6), in connection with the values of $\ddot{x}, \ddot{y}, \ddot{z}, X, Y, Z$ immediately after that instant. The values of $\ddot{x}, \ddot{y}, \ddot{z}$, thus determined, are of course the very quantities which we wish to obtain, since the acceleration of a point at a given instant does not denote anything different from its acceleration immediately after that instant.

For an example of a somewhat different class of cases, we may suppose that in a system, otherwise free, $\dot{x}$ cannot have a negative value. Such a condition does not seem to affect the possible values of δx, as naturally interpreted in a dynamical problem. Yet, if we should regard the value of δx in (7) as arbitrary, we should obtain

$$\ddot{x} = \frac{X}{m},$$

which might be erroneous. But if we regard δx as expressing a velocity of which the system, if at rest, would be capable (which is not a natural signification of the expression), we should have $\delta x \geqq 0$, which, with (7), gives

$$\ddot{x} \geqq \frac{X}{m}.$$

This is not incorrect, but it leaves the acceleration undetermined. If we should regard δx as denoting such a variation of the velocity as is possible for the system when it has its given velocity (this also is not a natural signification of the expression), formula (7) would give the correct value of $\ddot{x}$ except when $\dot{x} = 0$. In this case (which cannot be regarded as exceptional in a problem of this kind), we should have $\delta x \geqq 0$, which will leave $\ddot{x}$ undetermined, as before.

The application of formula (6), in problems of this kind, presents no difficulty. From the condition

$$\dot{x} \geqq 0,$$

we obtain, first, if $\dot{x} = 0$, $\ddot{x} \geqq 0$,

then, if $\dot{x} = 0$ and $\ddot{x} = 0$, $\delta\ddot{x} \geqq 0$,

which is the only limitation on the value of $\delta\ddot{x}$. With this condition, we deduce from (6) that either

$$\dot{x} = 0, \ \ddot{x} = 0, \text{ and } \ddot{x} \geqq \frac{X}{m};$$

or

$$\ddot{x} = \frac{X}{m}.$$

That is, if $\dot{x}=0$, $\ddot{x}$ has the greater of the values $\frac{X}{m}$ and 0; otherwise, $\ddot{x}=\frac{X}{m}$.

In cases of this kind also, in which the function which cannot exceed a certain value involves the velocities (with or without the coordinates), one may easily convince himself that formula (6) is always valid, and always sufficient to determine the accelerations with the aid of the conditions afforded by the constraints of the system.

But instead of examining such cases in detail, we shall proceed to consider the subject from a more general point of view.

Comparison of the New Formula with the Statical Principle of Virtual Velocities.—Case of Discontinuous Changes of Velocity.

Formula (1) has so far served as a point of departure. The general validity of this, the received form of the indeterminate equation of motion, being assumed, it has been shown that formula (6) will be valid and sufficient, even in cases in which both (1) and (7) fail. We now proceed to show that the statical principle of *virtual velocities*, when its real signification is carefully considered, leads directly to formula (6), or to an analogous formula for the determination of the discontinuous changes of velocity, when such occur. This will be the case even if we start with the usual analytical expression of the principle

$$\Sigma(X\delta x+Y\delta y+Z\delta z)\leqq 0, \tag{8}$$

to which, at first sight, formula (6) appears less closely related than (7). For the variations of the coordinates in this formula must be regarded as relating to differences between the configuration which the system has at a certain time, and which it will continue to have in case of equilibrium, and some other configuration which the system might be supposed to have at some subsequent time. These temporal relations are not indicated explicitly in the notation, and should not be, since the statical problem does not involve the time in any quantitative manner. But in a dynamical problem, in which we take account of the time, it is hardly natural to use δx, δy, δz in the same sense. In any problem in which x, y, z are regarded as functions of the time, δx, δy, δz are naturally understood to relate to differences between the configuration which the system has at a certain time, and some other configuration which it might (conceivably) have had at that time *instead of* that which it actually had.

Now when we suppose a point to have a certain position, specified by x, y, z, at a certain time, its position at that time is no longer a subject of hypothesis or of question. It is its future positions which

form the subject of inquiry. Its position in the immediate future is naturally specified by

$$x+\dot{x}dt+\tfrac{1}{2}\ddot{x}dt^2+\text{etc.},\quad y+\dot{y}dt+\tfrac{1}{2}\ddot{y}dt^2+\text{etc.},\quad z+\dot{z}dt+\tfrac{1}{2}\ddot{z}dt^2+\text{etc.},$$

and we may regard the variations of these expressions as corresponding to the δx, δy, δz of the statical problem. It is evidently sufficient to take account of the first term of these expressions of which the variation is not zero. Now, x, y, z, as has already been said, are to be regarded as constant. With respect to the terms containing $\dot{x}$, $\dot{y}$, $\dot{z}$, two cases are to be distinguished, according as there is, or is not, a finite change of velocity at the instant considered.

Let us first consider the most important case, in which there is no discontinuous change of velocity. In this case, $\dot{x}$, $\dot{y}$, $\dot{z}$ are not to be regarded as variable (by δ), and the variations of the above expressions are represented by

$$\tfrac{1}{2}\delta\ddot{x}\,dt^2,\quad \tfrac{1}{2}\delta\ddot{y}\,dt^2,\quad \tfrac{1}{2}\delta\ddot{z}\,dt^2,$$

which are, therefore, to be substituted for δx, δy, δz in the general formula of equilibrium (8) to adapt it to the conditions of a dynamical problem. By this substitution (in which the common factor $\frac{1}{2}dt^2$ may of course be omitted), and the addition of the terms expressing the reaction against acceleration, we obtain formula (6).

But if the circumstances are such that there is (or may be) a discontinuity in the values of $\dot{x}$, $\dot{y}$, $\dot{z}$ at the instant considered, it is necessary to distinguish the values of these expressions before and after the abrupt change. For this purpose, we may apply $\dot{x}$, $\dot{y}$, $\dot{z}$ to the original values, and denote the changed values by $\dot{x}+\Delta\dot{x}$, $\dot{y}+\Delta\dot{y}$, $\dot{z}+\Delta\dot{z}$. The value of x at a time very shortly subsequent to the instant considered, will be expressed by $x+(\dot{x}+\Delta\dot{x})dt+\text{etc.}$, in which we may regard $\Delta\dot{x}$ as subject to the variation denoted by δ. The variation of the expression is therefore $\delta\,\Delta\dot{x}\,dt$. Instead of $-m\ddot{x}$, which expresses the reaction against acceleration, we need in the present case $-m\Delta\dot{x}$ to express the reaction against the abrupt change of velocity. A reaction against such a change of velocity is, of course, to be regarded as infinite in intensity in comparison with reactions due to acceleration, and ordinary forces (such as cause acceleration) may be neglected in comparison. If, however, we conceive of the system as acted on by impulsive forces (i.e., such as have no finite duration, but are capable of producing finite changes of velocity, and are measured numerically by the discontinuities of velocity which they produce in the unit of mass), these forces should be combined with the reactions due to the discontinuities of velocity in the general formula which determines these discontinuities. If the impulsive forces are specified by X, Y, Z, the formula will be

$$[(\mathrm{X}-m\Delta\dot{x})\,\delta\Delta\dot{x}+(\mathrm{Y}-m\Delta\dot{y})\,\delta\Delta\dot{y}+(\mathrm{Z}-m\Delta\dot{z})\,\delta\Delta\dot{z})]\leqq 0. \qquad (9)$$

The reader will remark the strict analogy between this formula and (6), which would perhaps be more clearly exhibited if we should write $\frac{d\dot{x}}{dt}, \frac{d\dot{y}}{dt}, \frac{d\dot{z}}{dt}$ for $\ddot{x}, \ddot{y}, \ddot{z}$ in that formula.

But these formulæ may be established in a much more direct manner. For the formula (8), although for many purposes the most convenient expression of the principle of virtual velocities, is by no means the most convenient for our present purpose. As the usual name of the principle implies, it holds true of velocities as well as of displacements, and is perhaps more simple and more evident when thus applied.*

If we wish to apply the principle, thus understood, to a moving system so as to determine whether certain changes of velocity specified by $\Delta\dot{x}, \Delta\dot{y}, \Delta\dot{z}$ are those which the system will really receive at a given instant, the velocities to be multiplied into the forces and reactions in the most simple application of the principle are manifestly such as may be imagined to be compounded with the assumed velocities, and are therefore properly specified by $\delta\Delta\dot{x}, \delta\Delta\dot{y}, \delta\Delta\dot{z}$. The formula (9) may therefore be regarded as the most direct application of the principle of virtual velocities to discontinuous changes of velocity in a moving system.

In the case of a system in which there are no discontinuous changes of velocity, but which is subject to forces tending to produce accelerations, when we wish to determine whether certain accelerations, specified by $\ddot{x}, \ddot{y}, \ddot{z}$, are such as the system will really receive, it is evidently necessary to consider whether any possible variation of these accelerations is favored more than it is opposed by the forces

* Even in Statics, the principle of virtual *velocities*, as distinguished from that of virtual *displacements*, has a certain advantage in respect of its evidence. The demonstration of the principle in the first section of the *Mécanique Analytique*, if velocities had been considered instead of displacements, would not have been exposed to an objection, which has been expressed by M. Bertrand in the following words: "On a objecté, avec raison, à cette assertion de Lagrange l'example d'un point pesant en équilibre au sommet le plus élevé d'une courbe; il est évident qu'un déplacement infiniment petit le ferait descendre, et, pourtant, ce déplacement ne se produit pas." (*Mécanique Analytique*, troisème édition, tome 1, page 22, note de M. Bertrand.) The value of z (the height of the point above a horizontal plane) can certainly be diminished by a displacement of the point, but the value of $\dot{z}$ is not affected by any velocity given to the point.

The real difficulty in the consideration of displacements is that they are only possible at a time subsequent to that in which the system has the configuration to which the question of equilibrium relates. We may make the interval of time infinitely short, but it will always be difficult, in the establishing of fundamental principles, to treat a conception of this kind (relating to what is possible after an infinitesimal interval of time) with the same rigor as the idea of velocities or accelerations, which, in the cases to which (9) and (6) respectively relate, we may regard as communicated immediately to the system.

and reactions of the system. The formula (6) expresses a criterion of this kind in the most simple and direct manner. If we regard a force as a tendency to increase a quantity expressed by $\ddot{x}$, the product of the force by $\delta\ddot{x}$ is the natural measure of the extent to which this tendency is satisfied by an arbitrary variation of the accelerations. The principle expressed by the formula may not be very accurately designated by the words *virtual velocities*, but it certainly does not differ from the principle of virtual velocities (in the stricter sense of the term), more than this differs from that of virtual displacements,—a difference so slight that the distinction of the names is rarely insisted upon, and that it is often very difficult to tell which form of the principle is especially intended, even when the principle is enunciated or discussed somewhat at length.

But, although the formulæ (6) and (9) differ so little from the ordinary formulæ, they not only have a marked advantage in respect of precision and accuracy, but also may be more satisfactory to the mind, in that the changes considered (to which δ relates), are not so violently opposed to all the possibilities of the case as are those which are represented by the variations of the coordinates.* Moreover, as we shall see, they naturally lead to various important laws of motion.

Transformation of the New Formula.

Let us now consider some of the transformations of which our general formula (6) is capable. If we separate the terms containing

* It may have seemed to some readers of the *Mécanique Analytique*—a work of which the unity of method is one of the most striking characteristics, and that to which its universally recognized artistic merit is in great measure due—that the treatment of dynamical problems in that work is not entirely analogous to the treatment of statical problems. The statical question, whether a system will remain in equilibrium in a given configuration, is determined by Lagrange by considering all possible motions of the system and inquiring whether there is any reason why the system should take any one of them. A similar method in dynamics would be based upon a comparison of a proposed motion with all other motions of which the system is capable without violating its kinematical conditions. Instead of this, Lagrange virtually reduces the dynamical problem to a statical one, and considers, not the possible variations of the proposed motion, but the motions which would be possible if the system were at rest. This reduction of a given problem to a simpler one, which has already been solved, is a method which has its advantages, but it is not the characteristic method of the *Mécanique Analytique*. That which most distinguishes the plan of this treatise from the usual type is the direct application of the general principle to each particular case.

The point is perhaps of small moment, and may be differently regarded by others, but it is mentioned here because it was a feeling of this kind (whether justified or not) and the desire to express the formula of motion by means of a maximum or minimum condition, in which the conditions under which the maximum or minimum subsists should be such as the problem naturally affords (Gauss's principle of *least constraint* being at the time unknown to the present writer, and the conditions under which the minimum subsists in the principle of *least action* being such that that is hardly satisfactory as a fundamental principle), which led to the formulæ proposed in this paper.

the masses of the particles from those which contain the forces, we have

$$\Sigma(X\delta\ddot{x}+Y\delta\ddot{y}+Z\delta\ddot{z})-\Sigma[\tfrac{1}{2}m\delta(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)]\leqq 0, \tag{10}$$

or, if we write u for the acceleration of a particle,

$$\Sigma(X\delta\ddot{x}+Y\delta\ddot{y}+Z\delta\ddot{z})-\delta\Sigma(\tfrac{1}{2}mu^2)\leqq 0. \tag{11}$$

If, instead of terms of the form $X\delta x$, or in addition to such terms, equation (1) had contained terms of the form $P\delta p$, in which p denotes any quantity determined by the configuration of the system, it is evident that these would give terms of the form $P\delta\ddot{p}$ in (6), (10) and (11). For the considerations which justified the substitution of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ for δx, δy, δz in the usual formula were in no respect dependent upon the fact that x, y, z denote rectangular coordinates, but would apply equally to any other quantities which are determined by the configuration of the system.

Hence, if the moments of all the forces of the system are represented by the sum

$$\mathfrak{S}(Pdp),$$

the general formula of motion may be written

$$\mathfrak{S}(P\delta\ddot{p})-\delta\Sigma(\tfrac{1}{2}mu^2)\leqq 0 \tag{12}$$

If the forces admit of a force-function V, we have

$$\delta\ddot{V}-\delta\Sigma(\tfrac{1}{2}mu^2)\leqq 0,$$

or

$$\delta[\ddot{V}-\Sigma(\tfrac{1}{2}mu^2)]\leqq 0. \tag{13}$$

But if the forces are determined in any way whatever by the configuration and velocities of the system, with or without the time, X, Y, Z and P will be unaffected by the variation denoted by δ, and we may write the formula of motion in the form

$$\delta\Sigma(X\ddot{x}+Y\ddot{y}+Z\ddot{z}-\tfrac{1}{2}mu^2)\leqq 0, \tag{14}$$

or

$$\delta[\mathfrak{S}(P\ddot{p})-\Sigma(\tfrac{1}{2}mu^2)]\leqq 0. \tag{15}$$

If the forces are determined by the configuration alone, or the configuration and the time, $\delta\dot{X}=0$, $\delta\dot{Y}=0$, $\delta\dot{Z}=0$, $\delta\dot{P}=0$, and the general formula may be written

$$\delta\left[\frac{d}{dt}\Sigma(X\dot{x}+Y\dot{y}+Z\dot{z})-\Sigma(\tfrac{1}{2}mu^2)\right]\leqq 0, \tag{16}$$

or

$$\delta\left[\frac{d}{dt}\mathfrak{S}(P\dot{p})-\Sigma(\tfrac{1}{2}mu^2)\right]\leqq 0. \tag{17}$$

The quantity affected by δ in any one of the last five formulæ has not only a maximum value, but absolutely the greatest value consistent with the constraints of the system. This may be shown in reference to (15) by giving to $\ddot{p}$, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, contained explicitly or implicitly in the expression affected by δ, any possible finite

increments $\ddot{p}'$, $\ddot{x}'$, $\ddot{y}'$. $\ddot{z}'$, and subtracting the original value of the expression from the value thus modified. Now,

$$\begin{aligned}\mathfrak{S}[P(\ddot{p}+\ddot{p}')]-\Sigma[\tfrac{1}{2}m\{(\ddot{x}+\ddot{x}')^2+(\ddot{y}+\ddot{y}')^2+(\ddot{z}+\ddot{z}')^2\}]\\ -\mathfrak{S}(P\ddot{p})+\Sigma[\tfrac{1}{2}m(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)]\\ =\mathfrak{S}(P\ddot{p}')-\Sigma[m(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')]-\Sigma[\tfrac{1}{2}m(\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2)].\end{aligned}$$

But since $\ddot{p}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$ are proportional to and of the same sign with possible values of $\delta\ddot{p}$, $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$, we have, by the general formula of motion,

$$\mathfrak{S}(P\ddot{p}')-\Sigma[m(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')]\leqq 0.$$

The second member of the preceding equation is therefore negative. The first member is therefore negative, which proves the proposition with respect to (15). The demonstration is precisely the same with respect to (13) and (14), which may be regarded as particular cases of (15).

To show the same with regard to (16) and (17), we have only to observe that the quantities affected by δ in these formulæ differ from those affected by the same symbol in (14) and (15) only by the terms

$$\Sigma(\dot{X}\dot{x}+\dot{Y}\dot{y}+\dot{Z}\dot{z}) \quad \text{and} \quad \mathfrak{S}(\dot{P}\dot{p}),$$

which will not be affected by any change in the accelerations of the system.

When the forces are determined by the configuration (with or without the time), the principle may be enunciated as follows: The accelerations in the system are always such that the acceleration of the rate of work done by the forces diminished by one-half the sum of the products of the masses of the particles by the squares of their accelerations has the greatest possible value.

The formula (17), although in appearance less simple than (15), not only is more easily enunciated in words, but has the advantage that the quantity $\frac{d}{dt}\mathfrak{S}(P\dot{p})$ is entirely determined by the system with its forces and motions, which is not the case with $\mathfrak{S}(P\ddot{p})$. The value of the latter expression depends upon the manner in which we choose to represent the forces. For example, if a material point is revolving in a circle under the influence of a central force, we may write either $X\ddot{x}+Y\ddot{y}+Z\ddot{z}$ or $R\ddot{r}$ for $P\ddot{p}$, R and r denoting respectively the force and radius vector. Now $X\ddot{x}+Y\ddot{y}+Z\ddot{z}$ is manifestly unequal to $R\ddot{r}$. But $X\dot{x}+Y\dot{y}+Z\dot{z}$ is equal to $R\dot{r}$, and $\frac{d}{dt}(X\dot{x}+Y\dot{y}+Z\dot{z})$ is equal to $\frac{d}{dt}(R\dot{r})$.

It may not be without interest to see what shape our general formulæ will take in one of the most important cases of forces dependent upon the velocities. If a body which can be treated as a point is moving in a medium which presents a resistance expressed by

any function of the velocity, the terms due to that resistance in the general formula of motion may be expressed in the form

$$\delta\left[\phi(v)\frac{\dot{x}}{v}\ddot{x}+\phi(v)\frac{\dot{y}}{v}\ddot{y}+\phi(v)\frac{\dot{z}}{v}\ddot{z}\right],$$

where v denotes the velocity and $\phi(v)$ the resistance. But

$$\frac{\dot{x}\ddot{x}}{v}+\frac{\dot{y}\ddot{y}}{v}+\frac{\dot{z}\ddot{z}}{v}=\frac{dv}{dt}=\dot{v}.$$

The terms due to the resistance reduce, therefore, to

$$\delta[\phi(v)\dot{v}],$$

or,

$$\delta\frac{d}{dt}f(v),$$

where f denotes the primitive of the function denoted by ϕ.

Discontinuous Changes of Velocity.—Formula (9), which relates to discontinuous changes of velocity, is capable of similar transformations. If we set

$$w^2=\Delta\dot{x}^2+\Delta\dot{y}^2+\Delta\dot{z}^2,$$

the formula reduces to

$$\delta\Sigma(\mathrm{X}\Delta\dot{x}+\mathrm{Y}\Delta\dot{y}+\mathrm{Z}\Delta\dot{z}-\tfrac{1}{2}mw^2)\leqq 0, \tag{18}$$

where X, Y, Z are to be regarded as constant. If $\mathfrak{S}(\mathrm{P}dp)$ represents the sum of the moments of the impulsive forces, and we regard P as constant, we have

$$\delta[\mathfrak{S}(\mathrm{P}\Delta\dot{p})-\Sigma(\tfrac{1}{2}mw^2)]\leqq 0. \tag{19}$$

The expressions affected by δ in these formulæ have a greater value than they would receive from any other changes of velocity consistent with the constraints of the system.

Deduction of other Properties of Motion.

The principles which have been established furnish a convenient point of departure for the demonstration of various properties of motion relating to *maxima* and *minima*. We may obtain several such properties by considering how the accelerations of a system, at a given instant, will be modified by changes of the forces or of the constraints to which the system is subject. Let us suppose that the forces X, Y, Z of a system receive the increments X', Y', Z', in consequence of which, and of certain additional constraints, which do not produce any discontinuity in the velocities, the components of acceleration $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ receive the increments $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$. The expression

$$\Sigma\big[(X+X')(\ddot{x}+\ddot{x}')+(Y+Y')(\ddot{y}+\ddot{y}')+(Z+Z')(\ddot{z}+\ddot{z}') -\tfrac{1}{2}m\{(\ddot{x}+\ddot{x}')^2+(\ddot{y}+\ddot{y}')^2+(\ddot{z}+\ddot{z}')^2\}\big] \tag{20}$$

will be the greatest possible for any values of $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$ consistent with the constraints. But this expression may be divided into three parts,

$$\Sigma[(X+X')\ddot{x}+(Y+Y')\ddot{y}+(Z+Z')\ddot{z}-\tfrac{1}{2}m(\ddot{x}^2+\ddot{y}^2+\ddot{z}^2)], \tag{21}$$

$$\Sigma[X\ddot{x}'+Y\ddot{y}'+Z\ddot{z}'-m(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')], \tag{22}$$

and

$$\Sigma[X'\ddot{x}'+Y'\ddot{y}'+Z'\ddot{z}'-\tfrac{1}{2}m(\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2)]. \tag{23}$$

The first part is evidently constant with reference to variations of $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and may, therefore, be neglected. With respect to the second part we observe that by the general formula of the motion we have

$$\Sigma[X\delta\ddot{x}+Y\delta\ddot{y}+Z\delta\ddot{z}-m(\ddot{x}\delta\ddot{x}+\ddot{y}\delta\ddot{y}+\ddot{z}\delta\ddot{z})]=0$$

for all values of $\delta\ddot{x}$, $\delta\ddot{y}$, $\delta\ddot{z}$ which are possible and reversible before the addition of the new constraints. But values proportional to $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$, and of the same sign, are evidently consistent with the original constraints, and when the components of acceleration are altered to $\ddot{x}+\ddot{x}'$, $\ddot{y}+\ddot{y}'$, $\ddot{z}+\ddot{z}'$, variations of these quantities proportional to and of the same sign as $-\ddot{x}'$, $-\ddot{y}'$, $-\ddot{z}'$ are evidently consistent with the original constraints. Now if these latter variations were not possible before the accelerations were modified by the addition of the new forces and constraints, it must be that some constraint was then operative which afterwards ceased to be so. The expression (22) will, therefore, be equal to zero, provided only that all the constraints which were operative before the addition of the new forces and constraints, remain operative afterwards.* With this limitation, therefore, the expression (23) must have the greatest value consistent with the constraints. This principle may be expressed without reference to rectangular coordinates. If we write u' for the relative acceleration due to the additional forces and constraints, we have

$$u'^2=\ddot{x}'^2+\ddot{y}'^2+\ddot{z}'^2,$$

and expression (23) reduces to

$$\Sigma(X'\ddot{x}'+Y'\ddot{y}'+Z'\ddot{z}'-\tfrac{1}{2}mu'^2). \qquad (24)$$

If the sum of the moments of the additional forces which are considered is represented by $\mathfrak{S}(Qdq)$ (the q representing quantities determined by the configuration of the system), we have

$$\Sigma(X'\dot{x}+Y'\dot{y}+Z'\dot{z})=\mathfrak{S}(Q\dot{q}).$$

We may distinguish the values of $\frac{d^2q}{dt^2}$ immediately before and immediately after the application of the additional forces and constraints by the expressions $\ddot{q}$ and $\ddot{q}+\ddot{q}'$. With this understanding, we have, by differentiation of the preceding equation,

$$\Sigma[\dot{X}'\dot{x}+\dot{Y}'\dot{y}+\dot{Z}'\dot{z}+X'(\ddot{x}+\ddot{x}')+Y(\ddot{y}+\ddot{y}')+Z'(\ddot{z}+\ddot{z}')]$$
$$=\mathfrak{S}[\dot{Q}\dot{q}+Q(\ddot{q}+\ddot{q}')];$$

* As an illustration of the significance of this limitation, we may consider the condition afforded by the impenetrability of two bodies in contact. Let us suppose that if subject only to the original forces and constraints they would continue in contact, but that, under the influence of the additional forces and constraints, the contact will cease. The impenetrability of the bodies then ceases to be operative as a constraint. Such cases form an exception to the principle which is to be established. But there are no exceptions when all the original constraints are expressed by *equations*.

whence it appears that $\Sigma(X'\ddot{x}'+Y'\ddot{y}'+Z'\ddot{z}')$ differs from $\mathfrak{S}(Q\ddot{q}')$ only by quantities which are independent of the relative acceleration due to the additional forces and constraints. It follows that these relative accelerations are such as to make

$$\mathfrak{S}(Q\ddot{q}')-\Sigma(\tfrac{1}{2}mu'^2) \tag{25}$$

a maximum.

It will be observed that the condition which determines these relative accelerations is of precisely the same form as that which determines absolute accelerations.

An important case is that in which new constraints are added but no new forces. The relative accelerations are determined in this case by the condition that $\Sigma(\frac{1}{2}mu'^2)$ is a minimum. In any case of motion, in which finite forces do not act at points, lines or surfaces, we may first calculate the accelerations which would be produced if there were no constraints, and then determine the relative accelerations due to the constraints by the condition that $\Sigma(\frac{1}{2}mu'^2)$ is a minimum. This is Gauss's principle of *least constraint.**

Again, in any case of motion, we may suppose u to denote the acceleration which would be produced by the constraints alone, and u' the relative acceleration produced by the forces; we then have

$$\Sigma[m(\ddot{x}\ddot{x}'+\ddot{y}\ddot{y}'+\ddot{z}\ddot{z}')]=0,$$

whence, if we write u'' for the resultant or actual acceleration,

$$\Sigma(\tfrac{1}{2}mu^2)+\Sigma(\tfrac{1}{2}mu'^2)=\Sigma(\tfrac{1}{2}mu''^2).$$

Moreover, differentiating (25), we obtain

$$\mathfrak{S}(Q\delta\ddot{q}')-\Sigma[m(\ddot{x}'\delta\ddot{x}'+\ddot{y}'\delta\ddot{y}'+\ddot{z}'\delta\ddot{z}')]=0,$$

whence, since $\delta\ddot{q}'$, $\delta\ddot{x}'$, $\delta\ddot{y}'$, $\delta\ddot{z}'$ may have values proportional to $\ddot{q}'$, $\ddot{x}'$, $\ddot{y}'$, $\ddot{z}'$,

$$\mathfrak{S}(Q\ddot{q}')=2\Sigma(\tfrac{1}{2}mu'^2).$$

These relations are similar to those which exist with respect to *vis viva* and impulsive forces.

Particular Equations of Motion.

From the general formula (12), we may easily obtain particular equations which will express the laws of motion in a very general form.

Let $d\omega_1$, $d\omega_2$, etc. be infinitesimals (not necessarily complete differentials) the values of which are independent, and by means

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* This principle may be derived very directly from the general formula (6), or *vice versa*, for $\Sigma(\frac{1}{2}mu'^2)$ may be put in the form

$$\Sigma\left[\tfrac{1}{2}m\left\{\left(\ddot{x}-\frac{X}{m}\right)^2+\left(\ddot{y}-\frac{Y}{m}\right)^2+\left(\ddot{z}-\frac{Z}{m}\right)^2\right\}\right],$$

the variation of which, with the sign changed, is identical with the first member of (6).

of which we can perfectly define any infinitesimal change in the configuration of the system; and let

$$\dot{\omega}_1=\frac{d\omega_1}{dt},\quad \dot{\omega}_2=\frac{d\omega_2}{dt},\quad \text{etc.},$$

where $d\omega_1$, $d\omega_2$ are to be determined by the change in the configuration in the interval of time dt; and let

$$\ddot{\omega}_1=\frac{d\dot{\omega}_1}{dt},\quad \ddot{\omega}_2=\frac{d\dot{\omega}_2}{dt},\quad \text{etc.}$$

Also let $$U=\Sigma(\tfrac{1}{2}mu^2).$$

It is evident that U can be expressed in terms of $\dot{\omega}_1$, $\dot{\omega}_2$, etc., $\ddot{\omega}_1$, $\ddot{\omega}_2$, etc., and the quantities which express the configuration of the system, and that (since δ is used to denote a variation which does not affect the configuration or the velocities),

$$\delta U=\frac{dU}{d\ddot{\omega}_1}\delta\dot{\omega}_1+\frac{dU}{d\ddot{\omega}_2}\delta\dot{\omega}_2+\text{etc.}$$

Moreover, since the quantities p in the general formula are entirely determined by the configuration of the system

$$\dot{p}=\frac{dp}{d\omega_1}\dot{\omega}_1+\frac{dp}{d\omega_2}\dot{\omega}_2+\text{etc.},$$

where $\dfrac{dp}{d\omega_1}$ denotes the ratio of simultaneous values of dp and $d\omega_1$, when $d\omega_2$, etc. are equal to zero, and $\dfrac{dp}{d\omega_2}$, etc. are to be interpreted on the same principle. Multiplying by P, and taking the sum with respect to the several forces, we have

$$\mathfrak{S}(P\dot{p})=\Omega_1\dot{\omega}_1+\Omega_2\dot{\omega}_2+\text{etc.},$$

where $$\Omega_1=\mathfrak{S}\Big(P\frac{dp}{d\omega_1}\Big),\quad \Omega_2=\mathfrak{S}\Big(P\frac{dp}{d\omega_2}\Big),\quad \text{etc.}$$

If we differentiate with respect to t, and take the variation denoted by δ, we obtain

$$\mathfrak{S}(P\delta\ddot{p})=\Omega_1\delta\ddot{\omega}_1+\Omega_2\delta\ddot{\omega}_2+\text{etc.}$$

The general formula (12) is thus reduced to the form

$$\Big(\Omega_1-\frac{dU}{d\ddot{\omega}_1}\Big)\delta\ddot{\omega}_1+\Big(\Omega_2-\frac{dU}{d\ddot{\omega}_2}\Big)\delta\ddot{\omega}_2+\text{etc.}\gtrless 0. \tag{26}$$

If the forces have a potential V, we may write

$$\Big(\frac{dV}{d\omega_1}-\frac{dU}{d\ddot{\omega}_1}\Big)\delta\ddot{\omega}_1+\Big(\frac{dV}{d\omega_2}-\frac{dU}{d\ddot{\omega}_2}\Big)\delta\ddot{\omega}_2+\text{etc.}, \tag{27}$$

where $\dfrac{dV}{d\omega_1}$ denotes the ratio of dV and $d\omega_1$ when $d\omega_2$, etc. have the value zero, and the analogous expressions are to be interpreted on the same principle.

If the variations $\delta\omega_1$, $\delta\omega_2$, etc. are capable both of positive and of negative values, we must have

$$\frac{dU}{d\ddot{\omega}_1}=\Omega_1, \quad \frac{dU}{d\ddot{\omega}_2}=\Omega_2, \text{ etc.,} \tag{28}$$

or,

$$\frac{dU}{d\ddot{\omega}_1}=\frac{dV}{d\omega_1}, \quad \frac{dU}{d\ddot{\omega}_2}=\frac{dV}{d\omega_2}, \text{ etc.} \tag{29}$$

To illustrate the use of these equations in a case in which $d\omega_1$, $d\omega_2$, etc. are not exact differentials, we may apply them to the problem of the rotation of a rigid body of which one point is fixed. If $d\omega_1$, $d\omega_2$, $d\omega_3$ denote infinitesimal rotations about the principal axes which pass through the fixed point, Ω_1, Ω_2, Ω_3 will denote the moments of the impressed forces about these axes, and the value of U will be given by the formula

$$\begin{aligned}2U=&(a+b+c)(\dot{\omega}_1^2+\dot{\omega}_2^2+\dot{\omega}_3^2)^2-(\dot{\omega}_1^2+\dot{\omega}_2^2+\dot{\omega}_3^2)(a\dot{\omega}_1^2+b\dot{\omega}_2^2+c\dot{\omega}_3^2)\\&+2(b-c)\dot{\omega}_2\dot{\omega}_3\dot{\omega}_1+2(c-a)\dot{\omega}_3\dot{\omega}_1\dot{\omega}_2+2(a-b)\dot{\omega}_1\dot{\omega}_2\dot{\omega}_3\\&+(b+c)\ddot{\omega}_1^2+(c+a)\ddot{\omega}_2^2+(a+b)\ddot{\omega}_3^2,\end{aligned}$$

where a, b, and c are constants, $a+b$, $b+c$, $c+a$ being the *moments of inertia* about the three axes. Hence,

$$\frac{dU}{d\ddot{\omega}_1}=(b-c)\dot{\omega}_2\dot{\omega}_3+(b+c)\ddot{\omega}_1, \quad \frac{dU}{d\ddot{\omega}_2}=(c-a)\dot{\omega}_3\dot{\omega}_1+(c+a)\ddot{\omega}_2,$$

$$\frac{dU}{d\ddot{\omega}_3}=(a-b)\dot{\omega}_1\dot{\omega}_2+(a+b)\ddot{\omega}_3;$$

and the equations of motion are

$$\ddot{\omega}_1=\frac{(c-b)\dot{\omega}_2\dot{\omega}_3+\Omega_1}{c+b},$$

$$\ddot{\omega}_2=\frac{(a-c)\dot{\omega}_3\dot{\omega}_1+\Omega_2}{a+c},$$

$$\ddot{\omega}_3=\frac{(b-a)\dot{\omega}_1\dot{\omega}_2+\Omega_3}{b+a}.$$

II.

ON THE FUNDAMENTAL FORMULA OF STATISTICAL MECHANICS, WITH APPLICATIONS TO ASTRONOMY AND THERMODYNAMICS.

[*Proceedings of the American Association for the Advancement of Science*, vol. XXXIII. pp. 57, 58, 1884.]

(ABSTRACT.)

SUPPOSE that we have a great number of systems which consist of material points and are identical in character, but different in configuration and velocities, and in which the forces are determined by the configuration alone. Let the number of systems in which the coordinates and velocities lie severally between the following limits, viz., between

$$\begin{aligned} &x_1 \text{ and } x_1+dx_1,\\ &y_1 \text{ and } y_1+dy_1,\\ &z_1 \text{ and } z_1+dz_1,\\ &x_2 \text{ and } x_2+dx_2,\\ &\quad \text{etc.},\\ &\dot{x}_1 \text{ and } \dot{x}_1+d\dot{x}_1,\\ &\dot{y}_1 \text{ and } \dot{y}_1+d\dot{y}_1,\\ &\dot{z}_1 \text{ and } \dot{z}_1+d\dot{z}_1,\\ &\dot{x}_2 \text{ and } \dot{x}_2+d\dot{x}_2,\\ &\quad \text{etc.}, \end{aligned}$$

be denoted by

$$L\, dx_1\, dy_1\, dz_1\, dx_2 \text{ etc. } d\dot{x}_1\, d\dot{y}_1\, d\dot{z}_1\, d\dot{x}_2 \text{ etc.}$$

The manner in which the quantity L varies with the time is given by the equation

$$\frac{dL}{dt} = -\Sigma\left[\frac{dL}{dx}\dot{x}+\frac{dL}{d\dot{x}}\ddot{x}\right],$$

where t, x_1, y_1, z_1, x_2, etc., $\dot{x}_1$, $\dot{y}_1$, $\dot{z}_1$, $\dot{x}_2$, etc., are the independent variables, and the summation relates to all the coordinates.

The object of the paper is to establish this proposition (which is not claimed as new, but which has hardly received the recognition which it deserves) and to show its applications to astronomy and thermodynamics.

III.

ELEMENTS OF VECTOR ANALYSIS.

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(The fundamental principles of the following analysis are such as are familiar under a slightly different form to students of quaternions. The manner in which the subject is developed is somewhat different from that followed in treatises on quaternions, since the object of the writer does not require any use of the conception of the quaternion, being simply to give a suitable notation for those relations between vectors, or between vectors and scalars, which seem most important, and which lend themselves most readily to analytical transformations, and to explain some of these transformations. As a precedent for such a departure from quaternionic usage, Clifford's *Kinematic* may be cited. In this connection, the name of Grassmann may also be mentioned, to whose system the following method attaches itself in some respects more closely than to that of Hamilton.)

CHAPTER I.

CONCERNING THE ALGEBRA OF VECTORS.

Fundamental Notions.

1. *Definition.*—If anything has magnitude and direction, its magnitude and direction taken together constitute what is called a vector.

The numerical description of a vector requires three numbers, but nothing prevents us from using a single letter for its symbolical designation. An algebra or analytical method in which a single letter or other expression is used to specify a vector may be called a *vector algebra* or *vector analysis.*

Def.—As distinguished from vectors the real (positive or negative) quantities of ordinary algebra are called *scalars.**

As it is convenient that the form of the letter should indicate whether a vector or a scalar is denoted, we shall use the small Greek letters to denote vectors, and the small English letters to denote scalars. (The three letters, i, j, k, will make an exception, to be mentioned more particularly hereafter. Moreover, π will be used in its usual scalar sense, to denote the ratio of the circumference of a circle to its diameter.)

* The imaginaries of ordinary algebra may be called *biscalars*, and that which corresponds to them in the theory of vectors, *bivectors*. But we shall have no occasion to consider either of these. [See, however, footnote on p. 84.]

2. *Def.*—Vectors are said to be *equal* when they are the same both in direction and in magnitude. This equality is denoted by the ordinary sign, as $\alpha = \beta$. The reader will observe that this *vector equation* is the equivalent of three scalar equations.

A vector is said to be equal to zero, when its magnitude is zero. Such vectors may be set equal to one another, irrespectively of any considerations relating to direction.

3. Perhaps the most simple example of a vector is afforded by a directed straight line, as the line drawn from A to B. We may use the notation $\overline{AB}$ to denote this line as a vector, i.e., to denote its length and direction without regard to its position in other respects. The points A and B may be distinguished as the *origin* and the *terminus* of the vector. Since any magnitude may be represented by a length, any vector may be represented by a directed line; and it will often be convenient to use language relating to vectors, which refers to them as thus represented.

Reversal of Direction, Scalar Multiplication and Division.

4. The negative sign $(-)$ reverses the direction of a vector. (Sometimes the sign $+$ may be used to call attention to the fact that the vector has not the negative sign.)

Def.—A vector is said to be *multiplied* or *divided by a scalar* when its magnitude is multiplied or divided by the numerical value of the scalar and its direction is either unchanged or reversed according as the scalar is positive or negative. These operations are represented by the same methods as multiplication and division in algebra, and are to be regarded as substantially identical with them. The terms *scalar multiplication* and *scalar division* are used to denote multiplication and division by scalars, whether the quantity multiplied or divided is a scalar or a vector.

5. *Def.*—A unit vector is a vector of which the magnitude is unity.

Any vector may be regarded as the product of a positive scalar (the magnitude of the vector) and a unit vector.

The notation α_0 may be used to denote the magnitude of the vector α.

Addition and Subtraction of Vectors.

6. *Def.*—The *sum* of the vectors α, β, etc. (written $\alpha+\beta+$etc.) is the vector found by the following process. Assuming any point A, we determine successively the points B, C, etc., so that $\overline{AB}=\alpha$, $\overline{BC}=\beta$, etc. The vector drawn from A to the last point thus determined is the sum required. This is sometimes called the *geometrical* sum, to distinguish it from an *algebraic* sum or an *arithmetical* sum. It is also called the resultant, and α, β, etc. are called the components.

When the vectors to be added are all parallel to the same straight line, geometrical addition reduces to algebraic; when they have all the same direction, geometrical addition like algebraic reduces to arithmetical.

It may easily be shown that the value of a sum is not affected by changing the order of two consecutive terms, and therefore that it is not affected by any change in the order of the terms. Again, it is evident from the definition that the value of a sum is not altered by uniting any of its terms in brackets, as $\alpha+[\beta+\gamma]+\text{etc.}$, which is in effect to substitute the sum of the terms enclosed for the terms themselves among the vectors to be added. In other words, the commutative and associative principles of arithmetical and algebraic addition hold true of geometrical addition.

7. *Def.*—A vector is said to be subtracted when it is added after reversal of direction. This is indicated by the use of the sign $-$ instead of $+$.

8. It is easily shown that the distributive principle of arithmetical and algebraic multiplication applies to the multiplication of sums of vectors by scalars or sums of scalars, i.e.,

$$\begin{aligned}(m+n+\text{etc.})\,[\alpha+\beta+\text{etc.}] = {} & m\alpha+n\alpha+\text{etc.} \\ & +m\beta+n\beta+\text{etc.} \\ & +\text{etc.}\end{aligned}$$

9. *Vector Equations.*—If we have equations between sums and differences of vectors, we may transpose terms in them, multiply or divide by any scalar, and add or subtract the equations, precisely as in the case of the equations of ordinary algebra. Hence, if we have several such equations containing known and unknown vectors, the processes of elimination and reduction by which the unknown vectors may be expressed in terms of the known are precisely the same, and subject to the same limitations, as if the letters representing vectors represented scalars. This will be evident if we consider that in the multiplications incident to elimination in the supposed scalar equations the multipliers are the coefficients of the unknown quantities, or functions of these coefficients, and that such multiplications may be applied to the vector equations, since the coefficients are scalars.

10. *Linear relation of four vectors, Coordinates.*—If α, β, and γ are any given vectors not parallel to the same plane, any other vector ρ may be expressed in the form

$$\rho = a\alpha+b\beta+c\gamma.$$

If α, β, and γ are unit vectors, a, b, and c are the ordinary scalar components of ρ parallel to α, β, and γ. If $\rho=\overline{\text{OP}}$, ($\alpha$, β, γ being unit vectors), a, b, and c are the cartesian coordinates of the point P referred to axes through O parallel to α, β, and γ. When the values of these scalars are given, ρ is said to be given in terms of α, β, and γ.

It is generally in this way that the value of a vector is specified, viz., in terms of three known vectors. For such purposes of reference, a system of three mutually perpendicular vectors has certain evident advantages.

11. *Normal systems of unit vectors.*—The letters i, j, k are appropriated to the designation of a *normal system of unit vectors*, i.e., three unit vectors, each of which is at right angles to the other two and determined in direction by them in a perfectly definite manner. We shall always suppose that k is on the side of the i-j plane on which a rotation from i to j (through one right angle) appears counter-clockwise. In other words, the directions of i, j, and k are to be so determined that if they be turned (remaining rigidly connected with each other) so that i points to the east, and j to the north, k will point upward. When rectangular axes of X, Y, and Z are employed, their directions will be conformed to a similar condition, and i, j, k (when the contrary is not stated) will be supposed parallel to these axes respectively. We may have occasion to use more than one such system of unit vectors, just as we may use more than one system of coordinate axes. In such cases, the different systems may be distinguished by accents or otherwise.

12. *Numerical computation of a geometrical sum.*—If

$$\rho = a\alpha + b\beta + c\gamma,$$
$$\sigma = a'\alpha + b'\beta + c'\gamma,$$
$$\text{etc.},$$

then

$$\rho + \sigma + \text{etc.} = (a + a' + \text{etc.})\,\alpha + (b + b' + \text{etc.})\,\beta + (c + c' + \text{etc.})\,\gamma,$$

i.e., the coefficients by which a geometrical sum is expressed in terms of three vectors are the sums of the coefficients by which the separate terms of the geometrical sum are expressed in terms of the same three vectors.

Direct and Skew Products of Vectors.

13. *Def.*—The *direct product* of α and β (written $\alpha.\beta$) is the scalar quantity obtained by multiplying the product of their magnitudes by the cosine of the angle made by their directions.

14. *Def.*—The *skew product* of α and β (written $\alpha \times \beta$) is a vector function of α and β. Its magnitude is obtained by multiplying the product of the magnitudes of α and β by the sine of the angle made by their directions. Its direction is at right angles to α and β, and on that side of the plane containing α and β (supposed drawn from a common origin) on which a rotation from α to β through an arc of less than 180° appears counter-clockwise.

The direction of $\alpha \times \beta$ may also be defined as that in which an ordinary screw advances as it turns so as to carry α toward β.

Again, if α be directed toward the east, and β lie in the same horizontal plane and on the north side of α, $\alpha\times\beta$ will be directed upward.

15. It is evident from the preceding definitions that

$$\alpha.\beta=\beta.\alpha, \quad \text{and} \quad \alpha\times\beta=-\beta\times\alpha.$$

16. Moreover, $$[n\alpha].\beta=\alpha.[n\beta]=n[\alpha.\beta],$$

and $$[n\alpha]\times\beta=\alpha\times[n\beta]=n[\alpha\times\beta].$$

The brackets may therefore be omitted in such expressions.

17. From the definitions of No. 11 it appears that

$$i.i=j.j=k.k=1,$$

$$i.j=j.i=i.k=k.i=j.k=k.j=0,$$

$$i\times i=0, \quad j\times j=0, \quad k\times k=0,$$

$$i\times j=k, \quad j\times k=i, \quad k\times i=j,$$

$$j\times i=-k, \quad k\times j=-i, \quad i\times k=-j.$$

18. If we resolve β into two components β' and β'', of which the first is parallel and the second perpendicular to α, we shall have

$$\alpha.\beta=\alpha.\beta' \quad \text{and} \quad \alpha\times\beta=\alpha\times\beta''.$$

19. $$\alpha.[\beta+\gamma]=\alpha.\beta+\alpha.\gamma \text{ and } \alpha\times[\beta+\gamma]=\alpha\times\beta+\alpha\times\gamma.$$

To prove this, let $\sigma=\beta+\gamma$, and resolve each of the vectors β, γ, σ into two components, one parallel and the other perpendicular to α. Let these be β', β'', γ', γ'', σ', σ''. Then the equations to be proved will reduce by the last section to

$$\alpha.\sigma'=\alpha.\beta'+\alpha.\gamma' \quad \text{and} \quad \alpha\times\sigma''=\alpha\times\beta''+\alpha\times\gamma''.$$

Now since $\sigma=\beta+\gamma$ we may form a triangle in space, the sides of which shall be β, γ, and σ. Projecting this on a plane perpendicular to α, we obtain a triangle having the sides β'', γ'', and σ'', which affords the relation $\sigma''=\beta''+\gamma''$. If we pass planes perpendicular to α through the vertices of the first triangle, they will give on a line parallel to α segments equal to β', γ', σ'. Thus we obtain the relation $\sigma'=\beta'+\gamma'$. Therefore $\alpha.\sigma'=\alpha.\beta'+\alpha.\gamma'$, since all the cosines involved in these products are equal to unity. Moreover, if α is a unit vector, we shall evidently have $\alpha\times\sigma''=\alpha\times\beta''+\alpha\times\gamma''$, since the effect of the skew multiplication by α upon vectors in a plane perpendicular to α is simply to rotate them all 90° in that plane. But any case may be reduced to this by dividing both sides of the equation to be proved by the magnitude of α. The propositions are therefore proved.

20. Hence,

$$[\alpha+\beta].\gamma=\alpha.\gamma+\beta.\gamma, \qquad [\alpha+\beta]\times\gamma=\alpha\times\gamma+\beta\times\gamma,$$

$$[\alpha+\beta].[\gamma+\delta]=\alpha.\gamma+\alpha.\delta+\beta.\gamma+\beta.\delta,$$

$$[\alpha+\beta]\times[\gamma+\delta]=\alpha\times\gamma+\alpha\times\delta+\beta\times\gamma+\beta\times\delta;$$

and, in general, direct and skew products of sums of vectors may be expanded precisely as the products of sums in algebra, except that in skew products the order of the factors must not be changed without compensation in the sign of the term. If any of the terms in the factors have negative signs, the signs of the expanded product (when there is no change in the order of the factors) will be determined by the same rules as in algebra. It is on account of this analogy with algebraic products that these functions of vectors are called *products* and that other terms relating to multiplication are applied to them.

21. *Numerical calculation of direct and skew products.*—The properties demonstrated in the last two paragraphs (which may be briefly expressed by saying that the operations of direct and skew multiplication are distributive) afford the rule for the numerical calculation of a direct product, or of the components of a skew product, when the rectangular components of the factors are given numerically. In fact, if

$$\alpha = xi + yj + zk, \quad \text{and} \quad \beta = x'i + y'j + z'k;$$

$$\alpha . \beta = xx' + yy' + zz',$$

and $$\alpha \times \beta = (yz' - zy')i + (zx' - xz')j + (xy' - yx')k.$$

22. *Representation of the area of a parallelogram by a skew product.*—It will be easily seen that $\alpha \times \beta$ represents in magnitude the area of the parallelogram of which α and β (supposed drawn from a common origin) are the sides, and that it represents in direction the normal to the plane of the parallelogram on the side on which the rotation from α toward β appears counter-clockwise.

23. *Representation of the volume of a parallelopiped by a triple product.*—It will also be seen that $\alpha \times \beta . \gamma$* represents in numerical value the volume of the parallelopiped of which α, β, and γ (supposed drawn from a common origin) are the edges, and that the value of the expression is positive or negative according as γ lies on the side of the plane of α and β on which the rotation from α to β appears counter-clockwise, or on the opposite side.

24. Hence,

$$\begin{aligned} \alpha \times \beta . \gamma = \beta \times \gamma . \alpha = \gamma \times \alpha . \beta = \gamma . \alpha \times \beta = \alpha . \beta \times \gamma \\ = \beta . \gamma \times \alpha = -\beta \times \alpha . \gamma = -\gamma \times \beta . \alpha = -\alpha \times \gamma . \beta \\ = -\gamma . \beta \times \alpha = -\alpha . \gamma \times \beta = -\beta . \alpha \times \gamma . \end{aligned}$$

It will be observed that all the products of this type, which can be made with three given vectors, are the same in numerical value, and

* Since the sign $\times$ is only used between vectors, the skew multiplication in expressions of this kind is evidently to be performed first. In other words, the above expression must be interpreted as $[\alpha \times \beta] . \gamma$.

that any two such products are of the same or opposite character in respect to sign, according as the cyclic order of the letters is the same or different. The product vanishes when two of the vectors are parallel to the same line, or when the three are parallel to the same plane.

This kind of product may be called the scalar product of the three vectors. There are two other kinds of products of three vectors, both of which are vectors, viz., products of the type $(\alpha.\beta)\gamma$ or $\gamma(\alpha.\beta)$, and products of the type $\alpha\times[\beta\times\gamma]$ or $[\gamma\times\beta]\times\alpha$.

25. $i.j\times k=j.k\times i=k.i\times j=1. \qquad i.k\times j=k.j\times i=j.i\times k=-1.$

From these equations, which follow immediately from those of No. 17, the propositions of the last section might have been derived, viz., by substituting for α, β, and γ, respectively, expressions of the form $xi+yj+zk$, $x'i+y'j+z'k$, and $x''i+y''j+z''k$.* Such a method, which may be called *expansion in terms of* i, j, and k, will on many occasions afford very simple, although perhaps lengthy, demonstrations.

26. *Triple products containing only two different letters.*—The significance and the relations of $(\alpha.\alpha)\beta$, $(\alpha.\beta)\alpha$, and $\alpha\times[\alpha\times\beta]$ will be most evident, if we consider β as made up of two components, β' and β'', respectively parallel and perpendicular to α. Then

$$\beta=\beta'+\beta'',$$
$$(\alpha.\beta)\alpha=(\alpha.\beta')\alpha=(\alpha.\alpha)\beta',$$
$$\alpha\times[\alpha\times\beta]=\alpha\times[\alpha\times\beta'']=-(\alpha.\alpha)\beta''.$$

Hence, $$\alpha\times[\alpha\times\beta]=(\alpha.\beta)\alpha-(\alpha.\alpha)\beta.$$

27. *General relation of the vector products of three factors.*—In the triple product $\alpha\times[\beta\times\gamma]$ we may set

$$\alpha=l\beta+m\gamma+n\beta\times\gamma,$$

unless β and γ have the same direction. Then

$$\begin{aligned}\alpha\times[\beta\times\gamma]&=l\beta\times[\beta\times\gamma]+m\gamma\times[\beta\times\gamma]\\&=l(\beta.\gamma)\beta-l(\beta.\beta)\gamma-m(\gamma.\beta)\gamma+m(\gamma.\gamma)\beta\\&=(l\beta.\gamma+m\gamma.\gamma)\beta-(l\beta.\beta+m\gamma.\beta)\gamma.\end{aligned}$$

But $$l\beta.\gamma+m\gamma.\gamma=\alpha.\gamma, \text{ and } l\beta.\beta+m\gamma.\beta=\alpha.\beta.$$

Therefore $$\alpha\times[\beta\times\gamma]=(\alpha.\gamma)\beta-(\alpha.\beta)\gamma,$$

which is evidently true, when β and γ have the same directions. It may also be written

$$[\gamma\times\beta]\times\alpha=\beta(\gamma.\alpha)-\gamma(\beta.\alpha).$$

---

* The student who is familiar with the nature of determinants will not fail to observe that the triple product $\alpha.\beta\times\gamma$ is the determinant formed by the nine rectangular components of α, β, and γ, nor that the rectangular components of $\alpha\times\beta$ are determinants of the second order formed from the components of α and β. (See the last equation of No. 21.)

28. This principle may be used in the transformation of more complex products. It will be observed that its application will always simultaneously eliminate, or introduce, two signs of skew multiplication.

The student will easily prove the following identical equations, which, although of considerable importance, are here given principally as exercises in the application of the preceding formulæ.

29. $\alpha\times[\beta\times\gamma]+\beta\times[\gamma\times\alpha]+\gamma\times[\alpha\times\beta]=0.$

30. $$[\alpha\times\beta].[\gamma\times\delta]=(\alpha.\gamma)(\beta.\delta)-(\alpha.\delta)(\beta.\gamma).$$

31. $$[\alpha\times\beta]\times[\gamma\times\delta]=(\alpha.\gamma\times\delta)\beta-(\beta.\gamma\times\delta)\alpha$$
$$=(\alpha.\beta\times\delta)\gamma-(\alpha.\beta\times\gamma)\delta.$$

32. $$\alpha\times[\beta\times[\gamma\times\delta]]=(\alpha.\gamma\times\delta)\beta-(\alpha.\beta)\gamma\times\delta$$
$$=(\beta.\delta)\alpha\times\gamma-(\beta.\gamma)\alpha\times\delta.$$

33. $$[\alpha\times\beta].[\gamma\times\delta]\times[\epsilon\times\xi]=(\alpha.\beta\times\delta)(\gamma.\epsilon\times\xi)-(\alpha.\beta\times\gamma)(\delta.\epsilon\times\xi)$$
$$=(\alpha.\beta\times\epsilon)(\xi.\gamma\times\delta)-(\alpha.\beta\times\xi)(\epsilon.\gamma\times\delta)$$
$$=(\gamma.\delta\times\alpha)(\beta.\epsilon\times\xi)-(\gamma.\delta\times\beta)(\alpha.\epsilon\times\xi).$$

34. $[\alpha\times\beta].[\beta\times\gamma]\times[\gamma\times\alpha]=(\alpha.\beta\times\gamma)^2.$

35. The student will also easily convince himself that a product formed of any number of letters (representing vectors) combined in any possible way by scalar, direct, and skew multiplications may be reduced by the principles of Nos. 24 and 27 to a sum of products, each of which consists of scalar factors of the forms $\alpha.\beta$ and $\alpha.\beta\times\gamma$, with a single vector factor of the form α or $\alpha\times\beta$, when the original product is a vector.

36. *Elimination of scalars from vector equations.*—It has already been observed that the elimination of vectors from equations of the form

$$a\alpha+b\beta+c\gamma+d\delta+\text{etc.}=0$$

is performed by the same rule as the eliminations of ordinary algebra. (See No. 9.) But the elimination of scalars from such equations is at least formally different. Since a single vector equation is the equivalent of three scalar equations, we must be able to deduce from such an equation a scalar equation from which two of the scalars which appear in the original vector equation have been eliminated. We shall see how this may be done, if we consider the scalar equation

$$a\alpha.\lambda+b\beta.\lambda+c\gamma.\lambda+d\delta.\lambda+\text{etc.}=0,$$

which is derived from the above vector equation by direct multiplication by a vector λ. We may regard the original equation as the equivalent of the three scalar equations obtained by substituting for α, β, γ, δ, etc., their X-, Y-, and Z-components. The second equation would be derived from these by multiplying them respectively by

the X-, Y-, and Z-components of λ and adding. Hence the second equation may be regarded as the most general form of a scalar equation of the first degree in a, b, c, d, etc., which can be derived from the original vector equation or its equivalent three scalar equations. If we wish to have two of the scalars, as b and c, disappear, we have only to choose for λ a vector perpendicular to β and γ. Such a vector is $\beta\times\gamma$. We thus obtain

$$a\alpha.\beta\times\gamma+d\delta.\beta\times\gamma+\text{etc.}=0.$$

37. *Relations of four vectors.*—By this method of elimination we may find the values of the coefficients a, b, and c in the equation

$$\rho=a\alpha+b\beta+c\gamma, \tag{1}$$

by which any vector ρ is expressed in terms of three others. (See No. 10.) If we multiply *directly* by $\beta\times\gamma$, $\gamma\times\alpha$, and $\alpha\times\beta$, we obtain

$$\rho.\beta\times\gamma=a\alpha.\beta\times\gamma, \quad \rho.\gamma\times\alpha=b\beta.\gamma\times\alpha, \quad \rho.\alpha\times\beta=c\gamma.\alpha\times\beta; \tag{2}$$

whence

$$a=\frac{\rho.\beta\times\gamma}{\alpha.\beta\times\gamma}, \quad b=\frac{\rho.\gamma\times\alpha}{\alpha.\beta\times\gamma}, \quad c=\frac{\rho.\alpha\times\beta}{\alpha.\beta\times\gamma}. \tag{3}$$

By substitution of these values, we obtain the identical equation,

$$(\alpha.\beta\times\gamma)\,\rho=(\rho.\beta\times\gamma)\,\alpha+(\rho.\gamma\times\alpha)\,\beta+(\rho.\alpha\times\beta)\,\gamma. \tag{4}$$

(Compare No. 31.) If we wish the four vectors to appear symmetrically in the equation we may write

$$(\alpha.\beta\times\gamma)\,\rho-(\beta.\gamma\times\rho)\,\alpha+(\gamma.\rho\times\alpha)\,\beta-(\rho.\alpha\times\beta)\,\gamma=0. \tag{5}$$

If we wish to express ρ as a sum of vectors having directions perpendicular to the planes of α and β, of β and γ, and of γ and α, we may write

$$\rho=e\beta\times\gamma+f\gamma\times\alpha+g\alpha\times\beta. \tag{6}$$

To obtain the values of e, f, g, we multiply *directly* by α, by β, and by γ. This gives

$$e=\frac{\rho.\alpha}{\beta.\gamma\times\alpha}, \quad f=\frac{\rho.\beta}{\gamma.\alpha\times\beta}, \quad g=\frac{\rho.\gamma}{\alpha.\beta\times\gamma}. \tag{7}$$

Substituting these values we obtain the identical equation

$$(\alpha.\beta\times\gamma)\,\rho=(\rho.\alpha)\,\beta\times\gamma+(\rho.\beta)\,\gamma\times\alpha+(\rho.\gamma)\,\alpha\times\beta. \tag{8}$$

(Compare No. 32.)

38. *Reciprocal systems of vectors.*—The results of the preceding section may be more compactly expressed if we use the abbreviations

$$\alpha'=\frac{\beta\times\gamma}{\alpha.\beta\times\gamma}, \quad \beta'=\frac{\gamma\times\alpha}{\beta.\gamma\times\alpha}, \quad \gamma'=\frac{\alpha\times\beta}{\gamma.\alpha\times\beta}. \tag{1}$$

The identical equations (4) and (8) of the preceding number thus become

$$\rho=(\rho.\alpha')\alpha+(\rho.\beta')\beta+(\rho.\gamma')\gamma, \tag{2}$$

$$\rho=(\rho.\alpha)\alpha'+(\rho.\beta)\beta'+(\rho.\gamma)\gamma'. \tag{3}$$

We may infer from the similarity of these equations that the relations of α, β, γ, and α', β', γ' are reciprocal, a proposition which is easily proved directly. For the equations

$$\alpha=\frac{\beta'\times\gamma'}{\alpha'.\beta'\times\gamma'},\quad \beta=\frac{\gamma'\times\alpha'}{\beta'.\gamma'\times\alpha'},\quad \gamma=\frac{\alpha'\times\beta'}{\gamma'.\alpha'\times\beta'} \tag{4}$$

are satisfied identically by the substitution of the values of α', β', and γ' given in equations (1). (See Nos. 31 and 34.)

Def.—It will be convenient to use the term *reciprocal* to designate these relations, i.e., we shall say that three vectors are *reciprocals* of three others, when they satisfy relations similar to those expressed in equations (1) or (4).

With this understanding we may say:—

The coefficients by which any vector is expressed in terms of three other vectors are the direct products of that vector with the reciprocals of the three.

Among other relations which are satisfied by reciprocal systems of vectors are the following:

$$\alpha.\alpha'=\beta.\beta'=\gamma.\gamma'=1,$$

$$\alpha.\beta'=0,\quad \alpha.\gamma'=0,\quad \beta.\alpha'=0,\quad \beta.\gamma'=0,\quad \gamma.\alpha'=0,\quad \gamma.\beta'=0. \tag{5}$$

These nine equations may be regarded as defining the relations between α, β, γ and α', β', γ' as reciprocals.

$$(\alpha.\beta\times\gamma)(\alpha'.\beta'\times\gamma')=1. \tag{6}$$

(See No. 34.)

$$\alpha\times\alpha'+\beta\times\beta'+\gamma\times\gamma'=0. \tag{7}$$

(See No. 29.)

A system of three mutually perpendicular unit vectors is reciprocal to itself, and only such a system.

The identical equation

$$\rho=(\rho.i)i+(\rho.j)j+(\rho.k)k \tag{8}$$

may be regarded as a particular case of equation (2).

The system reciprocal to $\alpha\times\beta$, $\beta\times\gamma$, $\gamma\times\alpha$ is

$$\alpha'\times\beta',\quad \beta'\times\gamma',\quad \gamma'\times\alpha',$$

or

$$\frac{\alpha}{\alpha.\beta\times\gamma},\quad \frac{\beta}{\alpha.\beta\times\gamma},\quad \frac{\gamma}{\alpha.\beta\times\gamma}.$$

38*a*. If we multiply the identical equation (8) of No. 37 by $\sigma\times\tau$, we obtain the equation

$$\begin{aligned}(\alpha.\beta\times\gamma)(\rho.\sigma\times\tau)&=\alpha.\rho(\beta.\sigma\,\gamma.\tau-\beta.\tau\,\gamma.\sigma)\\&\quad+\beta.\rho(\gamma.\sigma\,\alpha.\tau-\gamma.\tau\,\alpha.\sigma)+\gamma.\rho(\alpha.\sigma\,\beta.\tau-\alpha.\tau\,\beta.\sigma),\end{aligned}$$

which is therefore identical. But this equation cannot subsist identically, unless

$$\begin{aligned}(\alpha.\beta\times\gamma)\sigma\times\tau&=\alpha(\beta.\sigma\,\gamma.\tau-\beta.\tau\,\gamma.\sigma)\\&\quad+\beta(\gamma.\sigma\,\alpha.\tau-\gamma.\tau\,\alpha.\sigma)+\gamma(\alpha.\sigma\,\beta.\tau-\alpha.\tau\,\beta.\sigma)\end{aligned}$$

is also an identical equation. (The reader will observe that in each of these equations the second member may be expressed as a determinant.)

From these transformations, with those already given, it follows that a product formed of any number of letters (representing vectors and scalars), combined in any possible way by scalar, direct, and skew multiplications, may be reduced to a sum of products, containing each the sign $\times$ once and only once, when the original product contains it an odd number of times, or entirely free from the sign, when the original product contains it an even number of times.

39. *Scalar equations of the first degree with respect to an unknown vector.*—It is easily shown that any scalar equation of the first degree with respect to an unknown vector ρ, in which all the other quantities are known, may be reduced to the form

$$\rho.\alpha=a,$$

in which α and a are known. (See No. 35.) Three such equations will afford the value of ρ (by equation (8) of No. 37, or equation (3) of No. 38), which may be used to eliminate ρ from any other equation either scalar or vector.

When we have four scalar equations of the first degree with respect to ρ, the elimination may be performed most symmetrically by substituting the values of $\rho.\alpha$, etc., in the equation

$$(\rho.\alpha)(\beta.\gamma\times\delta)-(\rho.\beta)(\gamma.\delta\times\alpha)+(\rho.\gamma)(\delta.\alpha\times\beta)-(\rho.\delta)(\alpha.\beta\times\gamma)=0,$$

which is obtained from equation (8) of No. 37 by multiplying directly by δ. It may also be obtained from equation (5) of No. 37 by writing δ for ρ, and then multiplying directly by ρ.

40. *Solution of a vector equation of the first degree with respect to the unknown vector.*—It is now easy to solve an equation of the form

$$\delta=a(\lambda.\rho)+\beta(\mu.\rho)+\gamma(\nu.\rho), \qquad (1)$$

where a, β, γ, δ, λ, μ, and ν represent known vectors. Multiplying directly by $\beta\times\gamma$, by $\gamma\times\alpha$, and by $\alpha\times\beta$, we obtain

$$\beta.\gamma\times\delta=(\beta.\gamma\times\alpha)(\lambda.\rho), \quad \gamma.\alpha\times\delta=(\gamma.\alpha\times\beta)(\mu.\rho),$$

$$\alpha.\beta\times\delta=(\alpha.\beta\times\gamma)(\nu.\rho);$$

or

$$\alpha'.\delta=\lambda.\rho, \quad \beta'.\delta=\mu.\rho, \quad \gamma'.\delta=\nu.\rho,$$

where α', β', γ' are the reciprocals of α, β, γ. Substituting these values in the identical equation

$$\rho=\lambda'(\lambda.\rho)+\mu'(\mu.\rho)+\nu'(\nu.\rho),$$

in which λ', μ', ν' are the reciprocals of λ, μ, ν (see No. 38), we have

$$\rho=\lambda'(\alpha'.\delta)+\mu'(\beta'.\delta)+\nu'(\gamma'.\delta), \qquad (2)$$

which is the solution required.

It results from the principle stated in No. 35, that any vector equation of the first degree with respect to ρ may be reduced to the form

$$\delta = a(\lambda.\rho) + \beta(\mu.\rho) + \gamma(\nu.\rho) + a\rho + \epsilon\times\rho.$$

But $$a\rho = a\lambda'(\lambda.\rho) + a\mu'(\mu.\rho) + a\nu'(\nu.\rho),$$

and $$\epsilon\times\rho = \epsilon\times\lambda'(\lambda.\rho) + \epsilon\times\mu'(\mu.\rho) + \epsilon\times\nu'(\nu.\rho),$$

where λ', μ', ν' represent, as before, the reciprocals of λ, μ, ν. By substitution of these values the equation is reduced to the form of equation (1), which may therefore be regarded as the most general form of a vector equation of the first degree with respect to ρ.

41. *Relations between two normal systems of unit vectors.*—If i, j, k, and i', j', k' are two normal systems of unit vectors, we have

$$\left.\begin{aligned} i' &= (i.i')i + (j.i')j + (k.i')k, \\ j' &= (i.j')i + (j.j')j + (k.j')k, \\ k' &= (i.k')i + (j.k')j + (k.k')k, \end{aligned}\right\} \quad (1)$$

and

$$\left.\begin{aligned} i &= (i.i')i' + (i.j')j' + (i.k')k', \\ j &= (j.i')i' + (j.j')j' + (j.k')k', \\ k &= (k.i')i' + (k.j')j' + (k.k')k'. \end{aligned}\right\} \quad (2)$$

(See equation (8) of No. 38.)

The nine coefficients in these equations are evidently the cosines of the nine angles made by a vector of one system with a vector of the other system. The principal relations of these cosines are easily deduced. By direct multiplication of each of the preceding equations with itself, we obtain six equations of the type

$$(i.i')^2 + (j.i')^2 + (k.i')^2 = 1. \quad (3)$$

By direct multiplication of equations (1) with each other, and of equations (2) with each other, we obtain six of the type

$$(i.i')(i.j') + (j.i')(j.j') + (k.i')(k.j') = 0. \quad (4)$$

By skew multiplication of equations (1) with each other, we obtain three of the type

$$k' = \{(j.i')(k.j') - (k.i')(j.j')\}i + \{(k.i')(i.j') - (i.i')(k.j')\}j + \{(i.i')(j.j') - (j.i')(i.j')\}k.$$

Comparing these three equations with the original three, we obtain nine of the type

$$i.k' = (j.i')(k.j') - (k.i')(j.j'). \quad (5)$$

Finally, if we equate the scalar product of the three right hand members of (1) with that of the three left hand members, we obtain

$$(i.i')(j.j')(k.k') + (i.j')(j.k')(k.i') + (i.k')(j.i')(k.j') - (k.i')(j.j')(i.k') - (k.j')(j.k')(i.i') - (k.k')(j.i')(i.j') = 1. \quad (6)$$

Equations (1) and (2) (if the expressions in the parentheses are supposed replaced by numerical values) represent the linear relations

which subsist between one vector of one system and the three vectors of the other system. If we desire to express the similar relations which subsist between two vectors of one system and two of the other, we may take the skew products of equations (1) with equations (2), after transposing all terms in the latter. This will afford nine equations of the type

$$(i.j')k' - (i.k')j' = (k.i')j - (j.i')k. \tag{7}$$

We may divide an equation by an indeterminate direct factor. [MS. note by author.]

CHAPTER II.

CONCERNING THE DIFFERENTIAL AND INTEGRAL CALCULUS OF VECTORS.

42. *Differentials of vectors.*—The *differential* of a vector is the *geometrical* difference of two values of that vector which differ infinitely little. It is itself a vector, and may make any angle with the vector differentiated. It is expressed by the same sign (d) as the differentials of ordinary analysis.

With reference to any fixed axes, the components of the differential of a vector are manifestly equal to the differentials of the components of the vector, i.e., if α, β, and γ are fixed unit vectors, and

$$\rho = x\alpha + y\beta + z\gamma,$$
$$d\rho = dx\,\alpha + dy\,\beta + dz\,\gamma.$$

43. *Differential of a function of several variables.*—The differential of a vector or scalar function of any number of vector or scalar variables is evidently the sum (geometrical or algebraic, according as the function is vector or scalar) of the differentials of the function due to the separate variation of the several variables.

44. *Differential of a product.*—The differential of a product of any kind due to the variation of a single factor is obtained by prefixing the sign of differentiation to that factor in the product. This is evidently true of differentials, since it will hold true even of finite differences.

45. From these principles we obtain the following identical equations:

$$d(\alpha+\beta) = d\alpha + d\beta, \tag{1}$$
$$d(n\alpha) = dn\,\alpha + n\,d\alpha, \tag{2}$$
$$d(\alpha.\beta) = d\alpha.\beta + \alpha.d\beta, \tag{3}$$
$$d[\alpha\times\beta] = d\alpha\times\beta + \alpha\times d\beta, \tag{4}$$
$$d(\alpha.\beta\times\gamma) = d\alpha.\beta\times\gamma + \alpha.d\beta\times\gamma + \alpha.\beta\times d\gamma, \tag{5}$$
$$d[(\alpha.\beta)\,\gamma] = (d\alpha.\beta)\,\gamma + (\alpha.d\beta)\,\gamma + (\alpha.\beta)\,d\gamma. \tag{6}$$

46. *Differential coefficient with respect to a scalar.*—The quotient obtained by dividing the differential of a vector due to the variation of any scalar of which it is a function by the differential of that scalar is called the differential coefficient of the vector with respect to the scalar, and is indicated in the same manner as the differential coefficients of ordinary analysis.

If we suppose the quantities occurring in the six equations of the last section to be functions of a scalar t, we may substitute $\frac{d}{dt}$ for d in those equations since this is only to divide all terms by the scalar dt.

47. *Successive differentiations.*—The differential coefficient of a vector with respect to a scalar is of course a finite vector, of which we may take the differential, or the differential coefficient with respect to the same or any other scalar. We thus obtain differential coefficients of the higher orders, which are indicated as in the scalar calculus.

A few examples will serve for illustration.

If ρ is the vector drawn from a fixed origin to a moving point at any time t, $\frac{d\rho}{dt}$ will be the vector representing the velocity of the point, and $\frac{d^2\rho}{dt^2}$ the vector representing its acceleration.

If ρ is the vector drawn from a fixed origin to any point on a curve, and s the distance of that point measured on the curve from any fixed point, $\frac{d\rho}{ds}$ is a unit vector, tangent to the curve and having the direction in which s increases; $\frac{d^2\rho}{ds^2}$ is a vector directed from a point on the curve to the center of curvature, and equal to the curvature; $\frac{d\rho}{ds} \times \frac{d^2\rho}{ds^2}$ is the normal to the osculating plane, directed to the side on which the curve appears described counter-clockwise about the center of curvature, and equal to the curvature. The tortuosity (or rate of rotation of the osculating plane, considered as positive when the rotation appears counter-clockwise as seen from the direction in which s increases) is represented by

$$\frac{\frac{d\rho}{ds} \cdot \frac{d^2\rho}{ds^2} \times \frac{d^3\rho}{ds^3}}{\frac{d^2\rho}{ds^2} \cdot \frac{d^2\rho}{ds^2}}.$$

48. *Integration of an equation between differentials.*—If t and u are two single-valued continuous scalar functions of any number of scalar or vector variables, and

$$dt = du,$$

then

$$t = u + a,$$

where a is a scalar constant.

Or, if τ and ω are two single-valued continuous vector functions of any number of scalar or vector variables, and

$$d\tau = d\omega,$$

then

$$\tau = \omega + a,$$

where a is a vector constant.

When the above hypotheses are not satisfied in general, but will be satisfied if the variations of the independent variables are confined within certain limits, then the conclusions will hold within those limits, provided that we can pass by continuous variation of the independent variables from any values within the limits to any other values within them, without transgressing the limits.

49. So far, it will be observed, all operations have been entirely analogous to those of the ordinary calculus.

Functions of Position in Space.

50. *Def.*—If u is any scalar function of position in space (i.e., any scalar quantity having continuously varying values in space), ∇u is the vector function of position in space which has everywhere the direction of the most rapid increase of u, and a magnitude equal to the rate of that increase per unit of length. ∇u may be called the *derivative* of u, and u, the *primitive* of ∇u.

We may also take any one of the Nos. 51, 52, 53 for the definition of ∇u.

51. If ρ is the vector defining the position of a point in space,

$$du = \nabla u . d\rho.$$

52.
$$\nabla u = i\frac{du}{dx} + j\frac{du}{dy} + k\frac{du}{dz}.$$

53.
$$\frac{du}{dx} = i.\nabla u, \quad \frac{du}{dy} = j.\nabla u, \quad \frac{du}{dz} = k.\nabla u.$$

54. *Def.*—If ω is a vector having continuously varying values in space,

$$\nabla.\omega = i.\frac{d\omega}{dx} + j.\frac{d\omega}{dy} + k.\frac{d\omega}{dz}, \tag{1}$$

and

$$\nabla\times\omega = i\times\frac{d\omega}{dx} + j\times\frac{d\omega}{dy} + k\times\frac{d\omega}{dz}. \tag{2}$$

$\nabla.\omega$ is called the *divergence* of ω and $\nabla\times\omega$ its *curl*.

If we set

$$\omega = Xi + Yj + Zk,$$

we obtain by substitution the equations

$$\nabla.\omega = \frac{dX}{dx} + \frac{dY}{dy} + \frac{dZ}{dz}$$

and

$$\nabla\times\omega = i\left(\frac{dZ}{dy} - \frac{dY}{dz}\right) + j\left(\frac{dX}{dz} - \frac{dZ}{dx}\right) + k\left(\frac{dY}{dx} - \frac{dX}{dy}\right),$$

which may also be regarded as defining $\nabla.\omega$ and $\nabla\times\omega$.

55. *Surface-integrals.*—The integral $\iint \omega . d\sigma$, in which $d\sigma$ represents an element of some surface, is called the surface-integral of ω for that surface. It is understood here and elsewhere, when a vector is said to represent a plane surface (or an element of surface which may be regarded as plane), that the magnitude of the vector represents the area of the surface, and that the direction of the vector represents that of the normal drawn toward the positive side of the surface. When the surface is defined as the boundary of a certain space, the outside of the surface is regarded as positive.

The surface-integral of any given space (i.e., the surface-integral of the surface bounding that space) is evidently equal to the sum of the surface-integrals of all the parts into which the original space may be divided. For the integrals relating to the surfaces dividing the parts will evidently cancel in such a sum.

The surface-integral of ω for a closed surface bounding a space dv infinitely small in all its dimensions is

$$\nabla . \omega \, dv.$$

This follows immediately from the definition of $\nabla\omega$, when the space is a parallelopiped bounded by planes perpendicular to i, j, k. In other cases, we may imagine the space—or rather a space nearly coincident with the given space and of the same volume dv—to be divided up into such parallelopipeds. The surface-integral for the space made up of the parallelopipeds will be the sum of the surface-integrals of all the parallelopipeds, and will therefore be expressed by $\nabla . \omega \, dv$. The surface-integral of the original space will have sensibly the same value, and will therefore be represented by the same formula. It follows that the value of $\nabla . \omega$ does not depend upon the system of unit vectors employed in its definition.

It is possible to attribute such a physical signification to the quantities concerned in the above proposition, as shall make it evident almost without demonstration. Let us suppose ω to represent a flux of any substance. The rate of decrease of the density of that substance at any point will be obtained by dividing the surface-integral of the flux for any infinitely small closed surface about the point by the volume enclosed. This quotient must therefore be independent of the form of the surface. We may define $\nabla . \omega$ as representing that quotient, and then obtain equation (1) of No. 54 by applying the general principle to the case of the rectangular parallelopiped.

56. *Skew surface-integrals.*—The integral $\iint d\sigma \times \omega$ may be called the skew surface-integral of ω. It is evidently a vector. For a closed surface bounding a space dv infinitely small in all dimensions this integral reduces to $\nabla \times \omega \, dv$, as is easily shown by reasoning like that of No. 55.

57. *Integration.*—If dv represents an element of any space, and $d\sigma$ an element of the bounding surface,

$$\iiint \nabla.\omega\, dv = \iint \omega.d\sigma.$$

For the first member of this equation represents the sum of the surface-integrals of all the elements of the given space. We may regard this principle as affording a means of integration, since we may use it to reduce a triple integral (of a certain form) to a double integral.

The principle may also be expressed as follows:

The surface-integral of any vector function of position in space for a closed surface is equal to the volume-integral of the divergence of that function for the space enclosed.

58. *Line-integrals.*—The integral $\int \omega.d\rho$, in which $d\rho$ denotes the element of a line, is called the *line-integral* of ω for that line. It is implied that one of the directions of the line is distinguished as positive. When the line is regarded as bounding a surface, that side of the surface will always be regarded as positive, on which the surface appears to be circumscribed counter-clockwise.

59. *Integration.*—From No. 51 we obtain directly

$$\int \nabla u.d\rho = u'' - u',$$

where the single and double accents distinguish the values relating to the beginning and end of the line.

In other words,—The line-integral of the derivative of any (continuous and single-valued) scalar function of position in space is equal to the difference of the values of the function at the extremities of the line. For a closed line the integral vanishes.

60. *Integration.*—The following principle may be used to reduce double integrals of a certain form to simple integrals.

If $d\sigma$ represents an element of any surface, and $d\rho$ an element of the bounding line,

$$\iint \nabla \times \omega.d\sigma = \int \omega.d\rho.$$

In other words,—The line-integral of any vector function of position in space for a closed line is equal to the surface-integral of the curl of that function for any surface bounded by the line.

To prove this principle, we will consider the variation of the line-integral which is due to a variation in the closed line for which the integral is taken. We have, in the first place,

$$\delta \int \omega.d\rho = \int \delta\omega.d\rho + \int \omega.\delta\, d\rho.$$

But

$$\omega.\delta\, d\rho = d(\omega.\delta\rho) - d\omega.\delta\rho.$$

Therefore, since $\int d(\omega.\delta\rho) = 0$ for a closed line,

$$\delta \int \omega.d\rho = \int \delta\omega.d\rho - \int d\omega.\delta\rho.$$

Now $$\delta\omega = \Sigma\left[\frac{d\omega}{dx}\delta x\right] = \Sigma\left[\frac{d\omega}{dx}(i.\delta\rho)\right],$$

and $$d\omega = \Sigma\left[\frac{d\omega}{dx}dx\right] = \Sigma\left[\frac{d\omega}{dx}(i.d\rho)\right],$$

where the summation relates to the coordinate axes and connected quantities. Substituting these values in the preceding equation, we get

$$\delta\int\omega.d\rho = \int\Sigma\left((i.\delta\rho)\left(\frac{d\omega}{dx}.d\rho\right) - (i.d\rho)\left(\frac{d\omega}{dx}.\delta\rho\right)\right),$$

or by No. 30,

$$\delta\int\omega.d\rho = \int\Sigma\left[i\times\frac{d\omega}{dx}\right].[\delta\rho\times d\rho] = \int\nabla\times\omega.[\delta\rho\times d\rho].$$

But $\delta\rho\times d\rho$ represents an element of the surface generated by the motion of the element $d\rho$, and the last member of the equation is the surface-integral of $\nabla\times\omega$ for the infinitesimal surface generated by the motion of the whole line. Hence, if we conceive of a closed curve passing gradually from an infinitesimal loop to any finite form, the differential of the line-integral of ω for that curve will be equal to the differential of the surface integral of $\nabla\times\omega$ for the surface generated: therefore, since both integrals commence with the value zero, they must always be equal to each other. Such a mode of generation will evidently apply to any surface closing any loop.

61. The line-integral of ω for a closed line bounding a plane surface $d\sigma$ infinitely small in all its dimensions is therefore

$$\nabla\times\omega.d\sigma.$$

This principle affords a definition of $\nabla\times\omega$ which is independent of any reference to coordinate axes. If we imagine a circle described about a fixed point to vary its orientation while keeping the same size, there will be a certain position of the circle for which the line-integral of ω will be a maximum, unless the line-integral vanishes for all positions of the circle. The axis of the circle in this position, drawn toward the side on which a positive motion in the circle appears counter-clockwise, gives the direction of $\nabla\times\omega$, and the quotient of the integral divided by the area of the circle gives the magnitude of $\nabla\times\omega$.

∇, $\nabla.$, *and* $\nabla\times$ *applied to Functions of Functions of Position.*

62. A constant scalar factor after ∇, $\nabla.$, or $\nabla\times$ may be placed before the symbol.

63. If $f(u)$ denotes any scalar function of u, and $f'(u)$ the derived function,

$$\nabla f(u) = f'(u)\nabla u.$$

64. If u or ω is a function of several scalar or vector variables which are themselves functions of the position of a single point, the value of ∇u or $\nabla.\omega$ or $\nabla\times\omega$ will be equal to the sum of the values obtained by making successively all but each one of these variables constant.

65. By the use of this principle we easily derive the following identical equations:

$$\nabla(t+u)=\nabla t+\nabla u. \tag{1}$$

$$\nabla.(\tau+\omega)=\nabla.\tau+\nabla.\omega. \qquad \nabla\times[\tau+\omega]=\nabla\times\tau+\nabla\times\omega. \tag{2}$$

$$\nabla(tu)=u\nabla t+t\nabla u. \tag{3}$$

$$\nabla.(u\omega)=\omega.\nabla u+u\nabla.\omega. \tag{4}$$

$$\nabla\times[u\omega]=u\nabla\times\omega-\omega\times\nabla u. \tag{5}$$

$$\nabla.[\tau\times\omega]=\omega.\nabla\times\tau-\tau.\nabla\times\omega. \tag{6}$$

The student will observe an analogy between these equations and the formulæ of multiplication. (In the last four equations the analogy appears most distinctly when we regard all the factors but one as constant.) Some of the more curious features of this analogy are due to the fact that the ∇ contains implicitly the vectors i, j and k, which are to be multiplied into the following quantities.

Combinations of the Operators ∇, $\nabla.$, *and* $\nabla\times$.

66. If u is any scalar function of position in space,

$$\nabla\times\nabla u=0,$$

as may be derived directly from the definitions of these operators.

67. Conversely, if ω is such a vector function of position in space that

$$\nabla\times\omega=0,$$

ω is the derivative of a scalar function of position in space. This will appear from the following considerations:

The line-integral $\int\omega.d\rho$ will vanish for any closed line, since it may be expressed as the surface-integral of $\nabla\times\omega$. (No. 60.) The line-integral taken from one given point P′ to another given point P″ is independent of the line between the points for which the integral is taken. (For, if two lines joining the same points gave different values, by reversing one we should obtain a closed line for which the integral would not vanish.) If we set u equal to this line-integral, supposing P″ to be variable and P′ to be constant in position, u will be a scalar function of the position of the point P″, satisfying the condition $du=\omega.d\rho$, or, by No. 51, $\nabla u=\omega$. There will evidently be an infinite number of functions satisfying this condition, which will differ from one another by constant quantities.

If the region for which $\nabla\times\omega=0$ is unlimited, these functions will be single-valued. If the region is limited, but acyclic,* the functions will still be single-valued and satisfy the condition $\nabla u=\omega$ within the same region. If the region is cyclic, we may determine functions satisfying the condition $\nabla u=\omega$ within the region, but they will not necessarily be single-valued.

68. If ω is any vector function of position in space, $\nabla.\nabla\times\omega=0$. This may be deduced directly from the definitions of No. 54.

The converse of this proposition will be proved hereafter.

69. If u is any scalar function of position in space, we have by Nos. 52 and 54

$$\nabla.\nabla u=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)u.$$

70. *Def.*—If ω is any vector function of position in space, we may define $\nabla.\nabla\omega$ by the equation

$$\nabla.\nabla\omega=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\omega,$$

the expression $\nabla.\nabla$ being regarded, for the present at least, as a single operator when applied to a vector. (It will be remembered that no meaning has been attributed to ∇ before a vector.) It should be noticed that if

$$\omega=i\mathrm{X}+j\mathrm{Y}+k\mathrm{Z},$$

$$\nabla.\nabla\omega=i\nabla.\nabla\mathrm{X}+j\nabla.\nabla\mathrm{Y}+k\nabla.\nabla\mathrm{Z},$$

that is, the operator $\nabla.\nabla$ applied to a vector affects separately its scalar components.

71. From the above definition with those of Nos. 52 and 54 we may easily obtain

$$\nabla.\nabla\omega=\nabla\nabla.\omega-\nabla\times\nabla\times\omega.$$

The effect of the operator $\nabla.\nabla$ is therefore independent of the directions of the axes used in its definition.

72. The expression $-\frac{1}{6}a^2\nabla.\nabla u$, where a is any infinitesimal scalar, evidently represents the excess of the value of the scalar function u

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* If every closed line within a given region can contract to a single point without breaking its continuity, or passing out of the region, the region is called *acyclic*, otherwise *cyclic*.

A cyclic region may be made acyclic by diaphragms, which must then be regarded as forming part of the surface bounding the region, each diaphragm contributing its own area twice to that surface. This process may be used to reduce many-valued functions of position in space, having single-valued derivatives, to single-valued functions.

When functions are mentioned or implied in the notation, the reader will always understand single-valued functions, unless the contrary is distinctly intimated, or the case is one in which the distinction is obviously immaterial. Diaphragms may be applied to bring functions naturally many-valued under the application of some of the following theorems, as Nos. 74 ff.

at the point considered above the average of its values at six points at the following vector distances: ai, $-ai$, aj, $-aj$, ak, $-ak$. Since the directions of i, j, and k are immaterial (provided that they are at right angles to each other), the excess of the value of u at the central point above its average value in a spherical surface of radius a constructed about that point as the center will be represented by the same expression, $-\frac{1}{6}a^2\nabla.\nabla u$.

Precisely the same is true of a vector function, if it is understood that the additions and subtractions implied in the terms *average* and *excess* are geometrical additions and subtractions.

Maxwell has called $-\nabla.\nabla u$ the *concentration* of u, whether u is scalar or vector. We may call $\nabla.\nabla u$ (or $\nabla.\nabla\omega$), which is proportioned to the excess of the average value of the function in an infinitesimal spherical surface above the value at the center, the *dispersion* of u (or ω).

Transformation of Definite Integrals.

73. From the equations of No. 65, with the principles of integration of Nos. 57, 59, and 60, we may deduce various transformations of definite integrals, which are entirely analogous to those known in the scalar calculus under the name of *integration by parts*. The following formulæ (like those of Nos. 57, 59, and 60) are written for the case of continuous values of the quantities (scalar and vector) to which the signs ∇, $\nabla.$, and $\nabla\times$ are applied. It is left to the student to complete the formulæ for cases of discontinuity in these values. The manner in which this is to be done may in each case be inferred from the nature of the formula itself. The most important discontinuities of scalars are those which occur at surfaces: in the case of vectors discontinuities at surfaces, at lines, and at points, should be considered.

74. From equation (3) we obtain

$$\int\nabla(tu).d\rho=t''u''-t'u'=\int u\nabla t.d\rho+\int t\nabla u.d\rho,$$

where the accents distinguish the quantities relating to the limits of the line-integrals. We are thus able to reduce a line-integral of the form $\int u\nabla t.d\rho$ to the form $-\int t\nabla u.d\rho$ with quantities free from the sign of integration.

75. From equation (5) we obtain

$$\iint\nabla\times(u\omega).d\sigma=\int u\omega.d\rho=\iint u\nabla\times\omega.d\sigma-\iint\omega\times\nabla u.d\sigma,$$

where, as elsewhere in these equations, the line-integral relates to the boundary of the surface-integral.

From this, by substitution of ∇t for ω, we may derive as a particular case

$$\iint\nabla u\times\nabla t.d\sigma=\int u\nabla t.d\rho=-\int t\nabla u.d\rho.$$

76. From equation (4) we obtain

$$\iiint \nabla.[u\omega]dv = \iint u\omega.d\sigma = \iiint \omega.\nabla u\, dv + \iiint u\nabla.\omega\, dv,$$

where, as elsewhere in these equations, the surface-integral relates to the boundary of the volume-integrals.

From this, by substitution of ∇t for ω, we derive as a particular case

$$\iiint \nabla t.\nabla u\, dv = \iint u\nabla t.d\sigma - \iiint u\nabla.\nabla t\, dv = \iint t\nabla u.d\sigma - \iiint t\nabla.\nabla u\, dv,$$

which is Green's Theorem. The substitution of $s\nabla t$ for ω gives the more general form of this theorem which is due to Thomson, viz.,

$$\begin{aligned}\iiint s\nabla t.\nabla u\, dv &= \iint us\nabla t.d\sigma - \iiint u\nabla.[s\nabla t]dv \\ &= \iint ts\nabla u.d\sigma - \iiint t\nabla.[s\nabla u]dv.\end{aligned}$$

77. From equation (6) we obtain

$$\iiint \nabla.[\tau\times\omega]dv = \iint \tau\times\omega.d\sigma = \iiint \omega.\nabla\times\tau\, dv - \iiint \tau.\nabla\times\omega\, dv.$$

A particular case is

$$\iiint \nabla u.\nabla\times\omega\, dv = \iint \omega\times\nabla u.d\sigma.$$

Integration of Differential Equations.

78. If throughout any continuous space (or in all space)

$$\nabla u = 0,$$

then throughout the same space

$$u = \text{constant}.$$

79. If throughout any continuous space (or in all space)

$$\nabla.\nabla u = 0,$$

and in any finite part of that space, or in any finite surface in or bounding it,

$$\nabla u = 0,$$

then throughout the whole space

$$\nabla u = 0, \text{ and } u = \text{constant}.$$

This will appear from the following considerations:

If $\nabla u = 0$ in any finite part of the space, u is constant in that part. If u is not constant throughout, let us imagine a sphere situated principally in the part in which u is constant, but projecting slightly into a part in which u has a greater value, or else into a part in which u has a less. The surface-integral of ∇u for the part of the spherical surface in the region where u is constant will have the value zero: for the other part of the surface, the integral will be either greater than zero, or less than zero. Therefore the whole surface-integral for the spherical surface will not have the value zero, which is required by the general condition, $\nabla.\nabla u = 0$.

Again, if $\nabla u = 0$ only in a surface in or bounding the space in

which $\nabla.\nabla u=0$, u will be constant in this surface, and the surface will be contiguous to a region in which $\nabla.\nabla u=0$ and u has a greater value than in the surface, or else a less value than in the surface. Let us imagine a sphere lying principally on the other side of the surface, but projecting slightly into this region, and let us particularly consider the surface-integral of ∇u for the small segment cut off by the surface $\nabla u=0$. The integral for that part of the surface of the segment which consists of part of the surface $\nabla u=0$ will have the value zero, the integral for the spherical part will have a value either greater than zero or else less than zero. Therefore the integral for the whole surface of the segment cannot have the value zero, which is demanded by the general condition, $\nabla.\nabla u=0$.

80. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla u=0,$$

and in all the bounding surfaces

$$u=\text{constant}=a,$$

and (in case the space extends to infinity) if at infinite distances within the space $u=a$,—then throughout the space

$$\nabla u=0, \text{ and } u=a.$$

For, if anywhere in the interior of the space ∇u has a value different from zero, we may find a point P where such is the case, and where u has a value b different from a,—to fix our ideas we will say *less*. Imagine a surface enclosing all of the space in which $u<b$. (This must be possible, since that part of the space does not reach to infinity.) The surface-integral of ∇u for this surface has the value zero in virtue of the general condition $\nabla.\nabla u=0$. But, from the manner in which the surface is defined, no part of the integral can be negative. Therefore no part of the integral can be positive, and the supposition made with respect to the point P is untenable. That the supposition that $b>a$ is untenable may be shown in a similar manner. Therefore the value of u is constant.

This proposition may be generalized by substituting the condition $\nabla.[t\nabla u]=0$ for $\nabla.\nabla u=0$, t denoting any positive (or any negative) scalar function of position in space. The conclusion would be the same, and the demonstration similar.

81. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla u=0,$$

and in all the bounding surfaces the normal component of ∇u vanishes, and at infinite distances within the space (if such there are) $r^2\frac{du}{dr}=0$,

where r denotes the distance from a fixed origin, then throughout the space

$$\nabla u = 0,$$

and in each continuous portion of the same

$$u = \text{constant}.$$

For, if anywhere in the space in question ∇u has a value different from zero, let it have such a value at a point P, and let u be there equal to b. Imagine a spherical surface about the above-mentioned origin as center, enclosing the point P, and with a radius r. Consider that portion of the space to which the theorem relates which is within the sphere and in which $u < b$. The surface integral of ∇u for this space is equal to zero in virtue of the general condition $\nabla.\nabla u = 0$. That part of the integral (if any) which relates to a portion of the spherical surface has a value numerically not greater than $4\pi r^2\left(\frac{du}{dr}\right)'$, where $\left(\frac{du}{dr}\right)'$ denotes the greatest numerical value of $\frac{du}{dr}$ in the portion of the spherical surface considered. Hence, the value of this part of the surface-integral may be made less (numerically) than any assignable quantity by giving to r a sufficiently great value. Hence, the other part of the surface-integral (viz., that relating to the surface in which $u = b$, and to the boundary of the space to which the theorem relates) may be given a value differing from zero by less than any assignable quantity. But no part of the integral relating to this surface can be negative. Therefore no part can be positive, and the supposition relative to the point P is untenable.

This proposition also may be generalized by substituting $\nabla.[t\nabla u] = 0$ for $\nabla.\nabla u = 0$, and $tr^2\frac{du}{dr} = 0$ for $r^2\frac{du}{dr} = 0$.

82. If throughout any continuous space (or in all space)

$$\nabla t = \nabla u,$$

then throughout the same space

$$t = u + \text{const.}$$

The truth of this and the three following theorems will be apparent if we consider the difference $t - u$.

83. If throughout any continuous space (or in all space)

$$\nabla.\nabla t = \nabla.\nabla u,$$

and in any finite part of that space, or in any finite surface in or bounding it,

$$\nabla t = \nabla u,$$

then throughout the whole space

$$\nabla t = \nabla u, \text{ and } t = u + \text{const.}$$

84. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla t = \nabla.\nabla u,$$

and in all the bounding surfaces

$$t = u,$$

and at infinite distances within the space (if such there are)

$$t = u,$$

then throughout the space

$$t = u.$$

85. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla.\nabla t = \nabla.\nabla u,$$

and in all the bounding surfaces the normal components of ∇t and ∇u are equal, and at infinite distances within the space (if such there are) $r^2\left(\frac{dt}{dr} - \frac{du}{dr}\right) = 0$, where r denotes the distance from some fixed origin, —then throughout the space

$$\nabla t = \nabla u,$$

and in each continuous part of which the space consists

$$t - u = \text{constant}.$$

86. If throughout any continuous space (or in all space)

$$\nabla\times\tau = \nabla\times\omega \quad \text{and} \quad \nabla.\tau = \nabla.\omega,$$

and in any finite part of that space, or in any finite surface in or bounding it,

$$\tau = \omega,$$

then throughout the whole space

$$\tau = \omega.$$

For, since $\nabla\times(\tau - \omega) = 0$, we may set $\nabla u = \tau - \omega$, making the space acyclic (if necessary) by diaphragms. Then in the whole space u is single-valued and $\nabla.\nabla u = 0$, and in a part of the space, or in a surface in or bounding it, $\nabla u = 0$. Hence throughout the space $\nabla u = \tau - \omega = 0$.

87. If throughout an aperiphractic* space contained within finite boundaries but not necessarily continuous

$$\nabla\times\tau = \nabla\times\omega \quad \text{and} \quad \nabla.\tau = \nabla.\omega,$$

and in all the bounding surfaces the tangential components of τ and ω are equal, then throughout the space

$$\tau = \omega.$$

It is evidently sufficient to prove this proposition for a continuous space. Setting $\nabla u = \tau - \omega$, we have $\nabla.\nabla u = 0$ for the whole space,

* If a space encloses within itself another space, it is called *periphractic*, otherwise *aperiphractic*.

and $u =$ constant for its boundary, which will be a single surface for a continuous aperiphractic space. Hence throughout the space

$$\nabla u = \tau - \omega = 0.$$

88. If throughout an acyclic space contained within finite boundaries but not necessarily continuous

$$\nabla \times \tau = \nabla \times \omega \quad \text{and} \quad \nabla . \tau = \nabla . \omega,$$

and in all the bounding surfaces the normal components of τ and ω are equal, then throughout the whole space

$$\tau = \omega.$$

Setting $\nabla u = \tau - \omega$, we have $\nabla . \nabla u = 0$ throughout the space, and the normal component of ∇u at the boundary equal to zero. Hence throughout the whole space $\nabla u = \tau - \omega = 0$.

89. If throughout a certain space (which need not be continuous, and which may extend to infinity)

$$\nabla . \nabla \tau = \nabla . \nabla \omega$$

and in all the bounding surfaces

$$\tau = \omega,$$

and at infinite distances within the space (if such there are)

$$\tau = \omega,$$

then throughout the whole space

$$\tau = \omega.$$

This will be apparent if we consider separately each of the scalar components of τ and ω.

Minimum Values of the Volume-integral $\iiint u\, \omega . \omega\, dv$. (*Thomson's Theorems.*)

90. Let it be required to determine for a certain space a vector function of position ω subject to certain conditions (to be specified hereafter), so that the volume-integral

$$\iiint u\, \omega . \omega\, dv$$

for that space shall have a minimum value, u denoting a given positive scalar function of position.

a. In the first place, let the vector ω be subject to the conditions that $\nabla . \omega$ is given within the space, and that the normal component of ω is given for the bounding surface. (This component must of course be such that the surface-integral of ω shall be equal to the volume-integral $\int \nabla . \omega\, dv$. If the space is not continuous, this must be true of each continuous portion of it. See No. 57.) The solution is that $\nabla \times (u\omega) = 0$, or more generally, that the line-integral of $u\omega$ for any closed curve in the space shall vanish.

The existence of the minimum requires that

$$\iiint u\,\omega.\delta\omega\,dv=0,$$

while $\delta\omega$ is subject to the limitation that

$$\nabla.\delta\omega=0,$$

and that the normal component of $\delta\omega$ at the bounding surface vanishes. To prove that the line-integral of $u\omega$ vanishes for any closed curve within the space, let us imagine the curve to be surrounded by an infinitely slender tube of normal section dz, which may be either constant or variable. We may satisfy the equation $\nabla.\delta\omega=0$ by making $\delta\omega=0$ outside of the tube, and $\delta\omega\,dz=\delta a\,\frac{d\rho}{ds}$ within it, δa denoting an arbitrary infinitesimal constant, ρ the position-vector, and ds an element of the length of the tube or closed curve. We have then

$$\iiint u\,\omega.\delta\omega\,dv=\int u\,\omega.\delta\omega\,dz\,ds=\int u\,\omega.d\rho\,\delta a=\delta a\int u\,\omega.d\rho=0,$$

whence $$\int u\,\omega.d\rho=0.$$ Q.E.D.

We may express this result by saying that $u\omega$ is the derivative of a single-valued scalar function of position in space. (See No. 67.)

If for certain parts of the surface the normal component of ω is not given for each point, but only the surface-integral of ω for each such part, then the above reasoning will apply not only to closed curves, but also to curves commencing and ending in such a part of the surface. The primitive of $u\omega$ will then have a constant value in each such part.

If the space extends to infinity and there is no special condition respecting the value of ω at infinite distances, the primitive of $u\omega$ will have a constant value at infinite distances within the space or within each separate continuous part of it.

If we except those cases in which the problem has no definite meaning because the data are such that the integral $\int u\,\omega.\omega\,dv$ must be infinite, it is evident that a minimum must always exist, and (on account of the quadratic form of the integral) that it is unique. That the conditions just found are sufficient to insure this minimum, is evident from the consideration that any allowable values of $\delta\omega$ may be made up of such values as we have supposed. Therefore, there will be one and only one vector function of position in space which satisfies these conditions together with those enumerated at the beginning of this number.

b. In the second place, let the vector ω be subject to the conditions that $\nabla\times\omega$ is given throughout the space, and that the tangential component of ω is given at the bounding surface. The solution is that

$$\nabla.[u\,\omega]=0,$$

and, if the space is periphractic, that the surface-integral of $u\omega$ vanishes for each of the bounding surfaces.

The existence of the minimum requires that

$$\iiint u\,\omega\,.\,\delta\omega\,dv = 0,$$

while $\delta\omega$ is subject to the conditions that

$$\nabla\times\delta\omega = 0,$$

and that the tangential component of $\delta\omega$ in the bounding surface vanishes. In virtue of these conditions we may set

$$\delta\omega = \nabla\delta q,$$

where δq is an arbitrary infinitesimal scalar function of position, subject only to the condition that it is constant in each of the bounding surfaces. (See No. 67.) By substitution of this value we obtain

$$\iiint u\,\omega\,.\,\nabla\delta q\,dv = 0,$$

or integrating by parts (No. 76)

$$\iint u\,\omega\,.\,d\sigma\,\delta q - \iiint \nabla\,.\,[u\,\omega]\delta q\,dv = 0.$$

Since δq is arbitrary in the volume-integral, we have throughout the whole space

$$\nabla\,.\,[u\,\omega] = 0\,;$$

and since δq has an arbitrary constant value in each of the bounding surfaces (if the boundary of the space consists of separate parts), we have for each such part

$$\iint u\,\omega\,.\,d\sigma = 0.$$

Potentials, Newtonians, Laplacians.

91. *Def.*—If u' is the scalar quantity of something situated at a certain point ρ', the *potential* of u' for any point ρ is a scalar function of ρ, defined by the equation

$$\text{pot}\,u' = \frac{u'}{[\rho'-\rho]_0},$$

and the Newtonian of u' for any point ρ is a vector function of ρ defined by the equation

$$\text{new}\,u' = \frac{\rho'-\rho}{[\rho'-\rho]_0^3}u'.$$

Again, if ω' is the vector representing the quantity and direction of something situated at the point ρ', the potential and the Laplacian of ω' for any point ρ are vector functions of ρ defined by the equations

$$\text{pot}\,\omega' = \frac{\omega'}{[\rho'-\rho]_0},$$

$$\text{lap}\,\omega' = \frac{\rho'-\rho}{[\rho'-\rho]_0^3}\times\omega'.$$

92. If u or ω is a scalar or vector function of position in space, we may write Pot u, New u, Pot ω, Lap ω for the volume-integrals of pot u', etc., taken as functions of ρ'; i.e., we may set

$$\text{Pot}\, u = \iiint \text{pot}\, u'\, dv' = \iiint \frac{u'}{[\rho' - \rho]_0}\, dv',$$

$$\text{New}\, u = \iiint \text{new}\, u'\, dv' = \iiint \frac{\rho' - \rho}{[\rho' - \rho]_0^3}\, u'\, dv',$$

$$\text{Pot}\, \omega = \iiint \text{pot}\, \omega'\, dv' = \iiint \frac{\omega'}{[\rho' - \rho]_0}\, dv',$$

$$\text{Lap}\, \omega = \iiint \text{lap}\, \omega'\, dv' = \iiint \frac{\rho' - \rho}{[\rho' - \rho]_0^3} \times \omega'\, dv',$$

where the ρ is to be regarded as constant in the integration. This extends over all space, or wherever the u' or ω' have any values other than zero. These integrals may themselves be called (integral) potentials, Newtonians, and Laplacians.

93. $$\frac{d\,\text{Pot}\, u}{dx} = \text{Pot}\frac{du}{dx}, \qquad \frac{d\,\text{Pot}\, \omega}{dx} = \text{Pot}\frac{d\omega}{dx}.$$

This will be evident with respect both to scalar and to vector functions, if we suppose that when we differentiate the potential with respect to x (thus varying the position of the point for which the potential is taken) each element of volume dv' in the implied integral remains fixed, *not in absolute position*, but in position relative to the point for which the potential is taken. This supposition is evidently allowable whenever the integration indicated by the symbol Pot tends to a definite limit when the limits of integration are indefinitely extended.

Since we may substitute y and z for x in the preceding formula, and since a constant factor of any kind may be introduced under the sign of integration, we have

$$\nabla\, \text{Pot}\, u = \text{Pot}\, \nabla u,$$
$$\nabla .\, \text{Pot}\, \omega = \text{Pot}\, \nabla .\omega,$$
$$\nabla \times \text{Pot}\, \omega = \text{Pot}\, \nabla \times \omega,$$
$$\nabla .\nabla\, \text{Pot}\, u = \text{Pot}\, \nabla .\nabla u,$$
$$\nabla .\nabla\, \text{Pot}\, \omega = \text{Pot}\, \nabla .\nabla \omega,$$

i.e., the symbols ∇, $\nabla .$, $\nabla \times$, $\nabla .\nabla$ may be applied indifferently before or after the sign Pot.

Yet a certain restriction is to be observed. When the operation of taking the (integral) potential does not give a definite finite value, the first members of these equations are to be regarded as entirely indeterminate, but the second members may have perfectly definite values. This would be the case, for example, if u or ω had a constant value throughout all space. It might seem harmless to set an indefinite expression equal to a definite, but it would be dangerous, since

we might with equal right set the indefinite expression equal to other definite expressions, and then be misled into supposing these definite expressions to be equal to one another. It will be safe to say that the above equations will hold, *provided that the potential of u or ω has a definite value.* It will be observed that whenever Pot u or Pot ω has a definite value *in general* (i.e., with the possible exception of certain points, lines, and surfaces),* the first members of all these equations will have definite values in general, and therefore the second members of the equation, being necessarily equal to the first members, when these have definite values, will also have definite values in general.

94. Again, whenever Pot u has a definite value we may write

$$\nabla \operatorname{Pot} u = \nabla \iiint \frac{u'}{r}\, dv' = \iiint \nabla \frac{1}{r}\, u'\, dv',$$

where r stands for $[\rho' - \rho]_0$. But

$$\nabla \frac{1}{r} = \frac{\rho' - \rho}{r^3},$$

whence $$\nabla \operatorname{Pot} u = \operatorname{New} u.$$

Moreover, New u will in general have a definite value, if Pot u has.

95. In like manner, whenever Pot ω has a definite value,

$$\nabla \times \operatorname{Pot} \omega = \nabla \times \iiint \frac{\omega'}{r}\, dv' = \iiint \nabla \times \frac{\omega'}{r}\, dv' = \iiint \nabla \frac{1}{r} \times \omega'\, dv'.$$

Substituting the value of $\nabla \frac{1}{r}$ given above we have

$$\nabla \times \operatorname{Pot} \omega = \operatorname{Lap} \omega.$$

Lap ω will have a definite value in general whenever Pot ω has.

96. Hence, with the aid of No. 93, we obtain

$$\nabla \times \operatorname{Lap} \omega = \operatorname{Lap} \nabla \times \omega,$$

$$\nabla . \operatorname{Lap} \omega = 0,$$

whenever Pot ω has a definite value.

97. By the method of No. 93 we obtain

$$\nabla . \operatorname{New} u = \nabla . \iiint \frac{\rho' - \rho}{r^3}\, u'\, dv' = \iiint \nabla u' . \frac{\rho' - \rho}{r^3}\, dv'.$$

To find the value of this integral, we may regard the point ρ, which is constant in the integration, as the center of polar coordinates. Then r becomes the radius vector of the point ρ', and we may set

$$dv' = r^2\, dq\, dr,$$

* Whenever it is said that a function of position in space has a definite value *in general*, this phrase is to be understood as explained above. The term definite is intended to exclude both indeterminate and infinite values.

where $r^2 dq$ is the element of a spherical surface having center at ρ and radius r. We may also set

$$\nabla u' . \frac{\rho' - \rho}{r} = \frac{du'}{dr}.$$

We thus obtain

$$\nabla . \text{New}\, u = \iiint \frac{du'}{dr}\, dq\, dr = 4\pi \int \frac{d\bar{u}'}{dr}\, dr = 4\pi \bar{u}'_{r=\infty} - 4\pi \bar{u}'_{r=0},$$

where $\bar{u}$ denotes the average value of u in a spherical surface of radius r about the point ρ as center.

Now if Pot u has in general a definite value, we must have $\bar{u}' = 0$ for $r = \infty$. Also, ∇. New u will have in general a definite value. For $r = 0$, the value of $\bar{u}'$ is evidently u. We have, therefore,

$$\nabla . \text{New}\, u = -4\pi u,$$

$$\nabla . \nabla \text{Pot}\, u = -4\pi u.*$$

98. If Pot ω has in general a definite value,

$$\begin{aligned} \nabla . \nabla \text{ Pot}\, \omega &= \nabla . \nabla \text{ Pot}\, [ui + vj + wk] \\ &= \nabla . \nabla \text{ Pot}\, ui + \nabla . \nabla \text{ Pot}\, vj + \nabla . \nabla \text{ Pot}\, wk \\ &= -4\pi ui - 4\pi vj - 4\pi wk \\ &= -4\pi \omega. \end{aligned}$$

Hence, by No. 71,

$$\nabla \times \nabla \times \text{Pot}\, \omega - \nabla \nabla . \text{Pot}\, \omega = 4\pi \omega.$$

That is,

$$\text{Lap}\, \nabla \times \omega - \text{New}\, \nabla . \omega = 4\pi \omega.$$

If we set

$$\omega_1 = \frac{1}{4\pi} \text{Lap}\, \nabla \times \omega, \quad \omega_2 = \frac{-1}{4\pi} \text{New}\, \nabla . \omega,$$

we have

$$\omega = \omega_1 + \omega_2,$$

where ω_1 and ω_2 are such functions of position that $\nabla . \omega_1 = 0$, and $\nabla \times \omega_2 = 0$. This is expressed by saying that ω_1 is *solenoidal*, and ω_2 *irrotational*. Pot ω_1 and Pot ω_2, like Pot ω, will have in general definite values.

It is worth while to notice that there is only one way in which a vector function of position in space having a definite potential can be thus divided into solenoidal and irrotational parts having definite potentials. For if $\omega_1 + \epsilon$, $\omega_2 - \epsilon$ are two other such parts,

$$\nabla . \epsilon = 0 \quad \text{and} \quad \nabla \times \epsilon = 0.$$

Moreover, Pot ϵ has in general a definite value, and therefore

$$\epsilon = \frac{1}{4\pi} \text{Lap}\, \nabla \times \epsilon - \frac{1}{4\pi} \text{New}\, \nabla . \epsilon = 0. \qquad \text{Q.E.D.}$$

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* Better thus: $\nabla . \nabla \text{ Pot}\, u = \iiint \frac{1}{r} \nabla . \nabla u\, dv = \iiint \nabla . \left(\frac{1}{r} \nabla u \right) dv - \iiint \nabla . \left(u \nabla \frac{1}{r} \right) dv + \iiint u \nabla . \nabla \frac{1}{r} dv$ $= -\iint u \nabla \frac{1}{r} . d\sigma = -4\pi u.$ [MS. note by author.]

99. To assist the memory of the student, some of the principal results of Nos. 93–98 may be expressed as follows:

Let ω_1 be any solenoidal vector function of position in space, ω_2 any irrotational vector function, and u any scalar function, satisfying the conditions that their potentials have in general definite values.

With respect to the solenoidal function ω_1, $\frac{1}{4\pi}\,\mathrm{Lap}$ and $\nabla\times$ are inverse operators; i.e.,

$$\frac{1}{4\pi}\,\mathrm{Lap}\,\nabla\times\omega_1=\nabla\times\frac{1}{4\pi}\,\mathrm{Lap}\,\omega_1=\omega_1.$$

Applied to the irrotational function ω_2, either of these operators gives zero; i.e.,

$$\mathrm{Lap}\,\omega_2=0,\quad \nabla\times\omega_2=0.$$

With respect to the irrotational function ω_2, or the scalar function u, $\frac{1}{4\pi}\,\mathrm{New}$ and $-\nabla.$ are inverse operators; i.e.,

$$-\frac{1}{4\pi}\,\mathrm{New}\,\nabla.\omega_2=\omega_2,\quad -\nabla.\frac{1}{4\pi}\,\mathrm{New}\,u=u.$$

Applied to the solenoidal function ω_1, the operator $\nabla.$ gives zero; i.e.

$$\nabla.\omega_1=0.$$

Since the most general form of a vector function having in general a definite potential may be written $\omega_1+\omega_2$, the effect of these operators on such a function needs no especial mention.

With respect to the solenoidal function ω_1, $\frac{1}{4\pi}\,\mathrm{Pot}$ and $\nabla\times\nabla\times$ are inverse operators; i.e.,

$$\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla\times\nabla\times\omega_1=\nabla\times\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla\times\omega_1=\nabla\times\nabla\times\frac{1}{4\pi}\,\mathrm{Pot}\,\omega_1=\omega_1.$$

With respect to the irrotational function ω_2, $\frac{1}{4\pi}\,\mathrm{Pot}$ and $-\nabla\nabla.$ are inverse operators; i.e.,

$$-\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla\nabla.\omega_2=-\nabla\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla.\omega_2=-\nabla\nabla.\frac{1}{4\pi}\,\mathrm{Pot}\,\omega_2=\omega_2.$$

With respect to any scalar or vector function having in general a definite potential $\frac{1}{4\pi}\,\mathrm{Pot}$ and $-\nabla.\nabla$ are inverse operators; i.e.,

$$-\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla.\nabla u=-\nabla.\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla u=-\nabla.\nabla\frac{1}{4\pi}\,\mathrm{Pot}\,u=u,$$

$$-\frac{1}{4\pi}\,\mathrm{Pot}\,\nabla.\nabla[\omega_1+\omega_2]=-\nabla.\nabla\frac{1}{4\pi}\,\mathrm{Pot}\,[\omega_1+\omega_2]=\omega_1+\omega_2.$$

With respect to the solenoidal function ω_1, $-\nabla.\nabla$ and $\nabla\times\nabla\times$ are equivalent; with respect to the irrotational function ω_2 $\nabla.\nabla$ and $\nabla\nabla$ are equivalent; i.e.,

$$-\nabla.\nabla\omega_1=\nabla\times\nabla\times\omega_1,\quad \nabla.\nabla\omega_2=\nabla\nabla.\omega_2.$$

100. *On the interpretation of the preceding formulæ.*—Infinite values of the quantity which occurs in a volume-integral as the coefficient of the element of volume will not necessarily make the value of the integral infinite, when they are confined to certain surfaces, lines, or points. Yet these surfaces, lines, or points may contribute a certain finite amount to the value of the volume-integral, which must be separately calculated, and in the case of surfaces or lines is naturally expressed as a surface- or line-integral. Such cases are easily treated by substituting for the surface, line, or point, a very thin shell, or filament, or a solid very small in all dimensions, within which the function may be supposed to have a very large value.

The only cases which we shall here consider in detail are those of surfaces at which the functions of position (u or ω) are discontinuous, and the values of ∇u, $\nabla\times\omega$, $\nabla.\omega$ thus become infinite. Let the function u have the value u_1 on the side of the surface which we regard as the negative, and the value u_2 on the positive side. Let $\Delta u = u_2 - u_1$. If we substitute for the surface a shell of very small thickness a, within which the value of u varies uniformly as we pass through the shell, we shall have $\nabla u = \nu \dfrac{\nabla u}{a}$ within the shell, ν denoting a unit normal on the positive side of the surface. The elements of volume which compose the shell may be expressed by $a[d\sigma]_0$, where $[d\sigma]_0$ is the magnitude of an element of the surface, $d\sigma$ being the vector element. Hence,

$$\nabla u\, dv = \nu\, \Delta u [d\sigma]_0 = \Delta u\, d\sigma.$$

Hence, when there are surfaces at which the values of u are discontinuous, the full value of Pot ∇u should always be understood as including the surface-integral

$$\iint \frac{\Delta u'}{[\rho' - \rho]_0}\, d\sigma'$$

relating to such surfaces. ($\Delta u'$ and $d\sigma'$ are accented in the formula to indicate that they relate to the point ρ'.)

In the case of a vector function which is discontinuous at a surface, the expressions $\nabla.\omega\, dv$ and $\nabla\times\omega\, dv$, relating to the element of the shell which we substitute for the surface of discontinuity, are easily transformed by the principle that these expressions are the direct and skew surface-integrals of ω for the element of the shell. (See Nos. 55, 56.) The part of the surface-integrals relating to the edge of the element may evidently be neglected, and we shall have

$$\nabla.\omega\, dv = \omega_2.d\sigma - \omega_1.d\sigma = \Delta\omega.d\sigma,$$

$$\nabla\times\omega\, dv = d\sigma\times\omega_2 - d\sigma\times\omega_1 = d\sigma\times\Delta\omega.$$

Whenever, therefore, ω is discontinuous at surfaces, the expressions Pot $\nabla.\omega$ and New $\nabla.\omega$ must be regarded as implicitly including the surface-integrals

$$\iint \frac{1}{[\rho'-\rho]_0}\Delta\omega'.d\sigma' \quad \text{and} \quad \iint \frac{\rho'-\rho}{[\rho'-\rho]_0^3}\Delta\omega'.d\sigma'$$

respectively, relating to such surfaces, and the expressions Pot $\nabla\times\omega$ and Lap $\nabla\times\omega$ as including the surface-integrals

$$\iint \frac{1}{[\rho'-\rho]_0}d\sigma'\times\Delta\omega' \quad \text{and} \quad \iint \frac{\rho'-\rho}{[\rho'-\rho]_0^3}\times[d\sigma'\times\Delta\omega']$$

respectively, relating to such surfaces.

101. We have already seen that if ω is the curl of any vector function of position, $\nabla.\omega=0$. (No. 68.) The converse is evidently true, whenever the equation $\nabla.\omega=0$ holds throughout all space, and ω has in general a definite potential; for then

$$\omega=\nabla\times\frac{1}{4\pi}\text{Lap}\,\omega.$$

Again, if $\nabla.\omega=0$ within any aperiphractic space A, contained within finite boundaries, we may suppose that space to be enclosed by a shell B having its inner surface coincident with the surface of A. We may imagine a function of position ω', such that $\omega'=\omega$ in A, $\omega'=0$ outside of the shell B, and the integral $\iiint\omega'.\omega'\,dv$ for B has the least value consistent with the conditions that the normal component of ω' at the outer surface is zero, and at the inner surface is equal to that of ω, and that in the shell $\nabla.\omega'=0$ (compare No. 90). Then $\nabla.\omega'=0$ throughout all space, and the potential of ω' will have in general a definite value. Hence,

$$\omega'=\nabla\times\frac{1}{4\pi}\text{Lap}\,\omega',$$

and ω will have the same value within the space A.

†102. *Def.*—If ω is a vector function of position in space, the *Maxwellian* * of ω is a scalar function of position defined by the equation

$$\text{Max}\,\omega=\iiint\frac{\rho'-\rho}{[\rho'-\rho]_0^3}.\omega'dv'.$$

(Compare No. 92.) From this definition the following properties are easily derived. It is supposed that the functions ω and u are such that their potentials have in general definite values.

$$\text{Max}\,\omega=\nabla.\text{Pot}\,\omega=\text{Pot}\,\nabla.\omega,$$
$$\nabla\,\text{Max}\,\omega=\nabla\nabla.\text{Pot}\,\omega=\text{New}\,\nabla.\omega,$$
$$\text{Max}\,\nabla u=-4\pi u,$$
$$4\pi\omega=\nabla\times\text{Lap}\,\omega-\nabla\,\text{Max}\,\omega.$$

* The frequent occurrence of the integral in Maxwell's *Treatise on Electricity and Magnetism* has suggested this name.

†[The foregoing portion of this paper was printed in 1881, the rest in 1884.]

If the values of Lap Lap ω, New Max ω, and Max New u are in general definite, we may add

$$4\pi \operatorname{Pot} \omega = \operatorname{Lap} \operatorname{Lap} \omega - \operatorname{New} \operatorname{Max} \omega,$$

$$4\pi \operatorname{Pot} u = -\operatorname{Max} \operatorname{New} u.$$

In other words: The Maxwellian is the divergence of the potential, $-\frac{\text{Max}}{4\pi}$ and ∇ are inverse operators for scalars and irrotational vectors, for vectors in general $-\frac{1}{4\pi}\nabla \operatorname{Max}$ is an operator which separates the irrotational from the solenoidal part. For scalars and irrotational vectors, $\frac{-1}{4\pi}\operatorname{Max}\operatorname{New}$ and $\frac{-1}{4\pi}\operatorname{New}\operatorname{Max}$ give the potential, for solenoidal vectors $\frac{1}{4\pi}\operatorname{Lap}\operatorname{Lap}$ gives the potential, for vectors in general $\frac{-1}{4\pi}\operatorname{New}\operatorname{Max}$ gives the potential of the irrotational part, and $\frac{1}{4\pi}\operatorname{Lap}\operatorname{Lap}$ the potential of the solenoidal part.

103. *Def.*—The following double volume-integrals are of frequent occurrence in physical problems. They are all scalar quantities, and none of them functions of position in space, as are the single volume-integrals which we have been considering. The integrations extend over all space, or as far as the expression to be integrated has values other than zero.

The *mutual potential*, or *potential product*, of two scalar functions of position in space is defined by the equation

$$\operatorname{Pot}(u, w) = \iiint\iiint \frac{uw'}{r}\, dv\, dv' = \iiint u \operatorname{Pot} w\, dv = \iiint w \operatorname{Pot} u\, dv.$$

In the double volume-integral, r is the distance between the two elements of volume, and u relates to dv as w' to dv'.

The *mutual potential*, or *potential product*, of two vector functions of position in space is defined by the equation

$$\operatorname{Pot}(\phi, \omega) = \iiint\iiint \frac{\phi \cdot \omega'}{r}\, dv\, dv' = \iiint \phi \,.\, \operatorname{Pot} \omega\, dv = \iiint \omega \,.\, \operatorname{Pot} \phi\, dv.$$

The *mutual Laplacian*, or *Laplacian product*, of two vector functions of position in space is defined by the equation

$$\operatorname{Lap}(\phi, \omega) = \iiint\iiint \omega \,.\, \frac{\rho' - \rho}{r^3} \times \phi'\, dv\, dv'$$

$$= \iiint \omega \,.\, \operatorname{Lap} \phi\, dv = \iiint \phi \,.\, \operatorname{Lap} \omega\, dv.$$

The *Newtonian product* of a scalar and a vector function of position in space is defined by the equation

$$\operatorname{New}(u, \omega) = \iiint\iiint \omega \,.\, \frac{\rho' - \rho}{r^3}\, u'\, dv\, dv' = \iiint \omega \,.\, \operatorname{New} u\, dv.$$

The *Maxwellian product* of a vector and a scalar function of position in space is defined by the equation

$$\text{Max}(\omega, u) = \iiint\iiint u \frac{\rho' - \rho}{r^3} . \omega' \, dv \, dv' = \iiint u \, \text{Max}\, \omega \, dv = -\text{New}(u, \omega).$$

It is of course supposed that u, w, ϕ, ω are such functions of position that the above expressions have definite values.

104. By No. 97,

$$4\pi u \, \text{Pot}\, w = -\nabla . \text{New}\, u \, \text{Pot}\, w = -\nabla . [\text{New}\, u \, \text{Pot}\, w] + \text{New}\, u . \text{New}\, w.$$

The volume-integral of this equation gives

$$4\pi \, \text{Pot}(u, w) = \iiint \text{New}\, u . \text{New}\, w \, dv,$$

if the integral

$$\iint d\sigma . \text{New}\, u \, \text{Pot}\, w,$$

for a closed surface, vanishes when the space included by the surface is indefinitely extended in all directions. This will be the case when everywhere outside of certain assignable limits the values of u and w are zero.

Again, by No. 102,

$$\begin{aligned} 4\pi\omega . \text{Pot}\, \phi &= \nabla \times \text{Lap}\, \omega . \text{Pot}\, \phi - \nabla \, \text{Max}\, \omega . \text{Pot}\, \phi \\ &= \nabla . [\text{Lap}\, \omega \times \text{Pot}\, \phi] + \text{Lap}\, \omega . \text{Lap}\, \phi \\ &\qquad - \nabla . [\text{Max}\, \omega \, \text{Pot}\, \phi] + \text{Max}\, \omega \, \text{Max}\, \phi. \end{aligned}$$

The volume-integral of this equation gives

$$4\pi \, \text{Pot}(\phi, \omega) = \iiint \text{Lap}\, \phi . \text{Lap}\, \omega \, dv + \iiint \text{Max}\, \phi \, \text{Max}\, \omega \, dv,$$

if the integrals

$$\iint d\sigma . \text{Lap}\, \omega \times \text{Pot}\, \phi, \qquad \iint d\sigma . \text{Pot}\, \phi \, \text{Max}\, \omega,$$

for a closed surface vanish when the space included by the surface is indefinitely extended in all directions. This will be the case if everywhere outside of certain assignable limits the values of ϕ and ω are zero.

CHAPTER III.

CONCERNING LINEAR VECTOR FUNCTIONS.

105. *Def.*—A vector function of a vector is said to be *linear*, when the function of the sum of any two vectors is equal to the sum of the functions of the vectors. That is, if

$$\text{func.}[\rho + \rho'] = \text{func.}[\rho] + \text{func.}[\rho']$$

for all values of ρ and ρ', the function is linear. In such cases it is easily shown that

$$\text{func.}[a\rho + b\rho' + c\rho'' + \text{etc.}] = a \, \text{func.}[\rho] + b \, \text{func.}[\rho'] + c \, \text{func.}[\rho''] + \text{etc.}$$

106. An expression of the form

$$\alpha\lambda.\rho+\beta\mu.\rho+\text{etc.}$$

evidently represents a linear function of ρ, and may be conveniently written in the form

$$\{\alpha\lambda+\beta\mu+\text{etc.}\}.\rho.$$

The expression $\rho.\alpha\lambda+\rho.\beta\mu+\text{etc.},$

or $\rho.\{\alpha\lambda+\beta\mu+\text{etc.}\},$

also represents a linear function of ρ, which is, in general, different from the preceding, and will be called its *conjugate*.

107. *Def.*—An expression of the form $\alpha\lambda$ or $\beta\mu$ will be called a *dyad*. An expression consisting of any number of dyads united by the signs + or − will be called a *dyadic binomial*, *trinomial*, etc., as the case may be, or more briefly, a *dyadic*. The latter term will be used so as to include the case of a single dyad. When we desire to express a dyadic by a single letter, the Greek capitals will be used, except such as are like the Roman, and also Δ and Σ. The letter I will also be used to represent a certain dyadic, to be mentioned hereafter.

Since any linear vector function may be expressed by means of a dyadic (as we shall see more particularly hereafter, see No. 110), the study of such functions, which is evidently of primary importance in the theory of vectors, may be reduced to that of dyadics.

108. *Def.*—Any two dyadics Φ and Ψ are equal,

when $\Phi.\rho=\Psi.\rho$ for all values of ρ,

or, when $\rho.\Phi=\rho.\Psi$ for all values of ρ,

or, when $\sigma.\Phi.\rho=\sigma.\Psi.\rho$ for all values of σ and of ρ.

The third condition is easily shown to be equivalent both to the first and to the second. The three conditions are therefore equivalent.

It follows that $\Phi=\Psi$, if $\Phi.\rho=\Psi.\rho$, or $\rho.\Phi=\rho.\Psi$, for three non-complanar values of ρ.

109. *Def.*—We shall call the vector $\Phi.\rho$ the (direct) product of Φ and ρ, the vector $\rho.\Phi$ the (direct) product of ρ and Φ, and the scalar $\sigma.\Phi.\rho$ the (direct) product of σ, Φ, and ρ.

In the combination $\Phi.\rho$, we shall say that Φ is used as a *prefactor*, in the combination $\rho.\Phi$, as a *postfactor*.

110. If τ is any linear function of ρ, and for $\rho=i$, $\rho=j$, $\rho=k$, the values of τ are respectively α, β, and γ, we may set

$$\tau=\{\alpha i+\beta j+\gamma k\}.\rho,$$

and also

$$\tau=\rho.\{i\alpha+j\beta+k\gamma\}.$$

Therefore, any linear function may be expressed by a dyadic as prefactor and also by a dyadic as postfactor.

111. *Def.*—We shall say that a dyadic is multiplied by a scalar, when one of the vectors of each of its component dyads is multiplied by that scalar. It is evidently immaterial to which vector of any dyad the scalar factor is applied. The product of the dyadic Φ and the scalar a may be written either $a\Phi$ or Φa. The minus sign before a dyadic reverses the signs of all its terms.

112. The sign + in a dyadic, or connecting dyadics, may be regarded as expressing addition, since the combination of dyads and dyadics with this sign is subject to the laws of association and commutation.

113. The combination of vectors in a dyad is evidently distributive. That is,

$$[\alpha+\beta+\text{etc.}][\lambda+\mu+\text{etc.}]=a\lambda+a\mu+\beta\lambda+\beta\mu+\text{etc.}$$

We may therefore regard the dyad as a kind of product of the two vectors of which it is formed. Since this kind of product is not commutative, we shall have occasion to distinguish the factors as *antecedent* and *consequent*.

114. Since any vector may be expressed as a sum of i, j, and k with scalar coefficients, every dyadic may be reduced to a sum of the nine dyads

$$ii,\ ij,\ ik,\ ji,\ jj,\ jk,\ ki,\ kj,\ kk,$$

with scalar coefficients. Two such sums cannot be equal according to the definitions of No. 108, unless their coefficients are equal each to each. Hence dyadics are equal only when their equality can be deduced from the principle that the operation of forming a dyad is a distributive one.

On this account, we may regard the dyad as the most general form of product of two vectors. We shall call it the indeterminate product. The complete determination of a single dyad involves five independent scalars, of a dyadic, nine.

115. It follows from the principles of the last paragraph that if

$$\Sigma a\beta=\Sigma\kappa\lambda,$$

then

$$\Sigma a\times\beta=\Sigma\kappa\times\lambda,$$

and

$$\Sigma a.\beta=\Sigma\kappa.\lambda.$$

In other words, the vector and the scalar obtained from a dyadic by insertion of the sign of skew or direct multiplication in each dyad are both independent of the particular form in which the dyadic is expressed.

We shall write $\Phi_\times$ and Φ_s to indicate the vector and the scalar thus obtained.

$$\Phi_\times=(j.\Phi.k-k.\Phi.j)i+(k.\Phi.i-i.\Phi.k)j+(i.\Phi.j-j.\Phi.i)k,$$

$$\Phi_s=i.\Phi.i+j.\Phi.j+k.\Phi.k,$$

as is at once evident, if we suppose Φ to be expanded in terms of ii, ij, etc.

116. *Def.*—The (*direct*) *product* of two dyads (indicated by a dot) is the dyad formed of the first and last of the four factors, multiplied by the direct product of the second and third. That is,

$$\{\alpha\beta\}.\{\gamma\delta\} = \alpha\,\beta.\gamma\,\delta = \beta.\gamma\,\alpha\delta.$$

The (direct) product of two dyadics is the sum of all the products formed by prefixing a term of the first dyadic to a term of the second. Since the direct product of one dyadic with another is a dyadic, it may be multiplied in the same way by a third, and so on indefinitely. This kind of multiplication is evidently associative, as well as distributive. The same is true of the direct product of a series of factors of which the first and the last are either dyadics or vectors, and the other factors are dyadics. Thus the values of the expressions

$$\alpha.\Phi.\Theta.\Psi.\beta, \quad \alpha.\Phi.\Theta, \quad \Phi.\Theta.\Psi.\beta, \quad \Phi.\Theta.\Psi$$

will not be affected by any insertion of parentheses. But this kind of multiplication is not commutative, except in the case of the direct product of two vectors.

117. *Def.*—The expressions $\Phi\times\rho$ and $\rho\times\Phi$ represent dyadics which we shall call the *skew* products of Φ and ρ. If

$$\Phi = \alpha\lambda + \beta\mu + \text{etc.},$$

these skew products are defined by the equations

$$\Phi\times\rho = \alpha\,\lambda\times\rho + \beta\,\mu\times\rho + \text{etc.},$$
$$\rho\times\Phi = \rho\times\alpha\,\lambda + \rho\times\beta\,\mu + \text{etc.}$$

It is evident that

$$\{\rho\times\Phi\}.\Psi = \rho\times\{\Phi.\Psi\}, \qquad \Psi.\{\Phi\times\rho\} = \{\Psi.\Phi\}\times\rho,$$
$$\{\rho\times\Phi\}.\alpha = \rho\times[\Phi.\alpha], \qquad \alpha.\{\Phi\times\rho\} = [\alpha.\Phi]\times\rho,$$
$$\{\rho\times\Phi\}\times\alpha = \rho\times\{\Phi\times\alpha\}.$$

We may therefore write without ambiguity

$$\rho\times\Phi.\Psi, \quad \Psi.\Phi\times\rho, \quad \rho\times\Phi.\alpha, \quad \alpha.\Phi\times\rho, \quad \rho\times\Phi.\alpha.$$

This may be expressed a little more generally by saying that the associative principle enunciated in No. 116 may be extended to cases in which the initial or final vectors are connected with the other factors by the sign of skew multiplication.

Moreover,

$$\alpha.\rho\times\Phi = [\alpha\times\rho].\Phi \text{ and } \Phi\times\rho.\alpha = \Phi.[\rho\times\alpha].$$

These expressions evidently represent vectors. So

$$\Psi.\{\rho\times\Phi\} = \{\Psi\times\rho\}.\Phi.$$

These expressions represent dyadics. The braces cannot be omitted without ambiguity.

118. Since all the antecedents or all the consequents in any dyadic may be expressed in parts of any three non-complanar vectors, and since the sum of any number of dyads having the same antecedent or the same consequent may be expressed by a single dyad, it follows that any dyadic may be expressed as the sum of three dyads, and so, that either the antecedents or the consequents shall be any desired non-complanar vectors, but only in one way when either the antecedents or the consequents are thus given.

In particular, the dyadic

$$\begin{aligned} &ai\,i+bi\,j\ +ci\,k \\ +&a'j\,i\ +b'j\,j\ +c'j\,k \\ +&a''k\,i+b''k\,j+c''k\,k, \end{aligned}$$

which may for brevity be written

$$\begin{Bmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{Bmatrix},$$

is equal to

$$a\,i+\beta\,j+\gamma\,k,$$

where

$$\begin{aligned} \alpha &= a\,i+a'j+a''k, \\ \beta &= b\,i+b'j+b''k, \\ \gamma &= c\,i+c'j+c''k, \end{aligned}$$

and to

$$i\lambda+j\mu+k\nu,$$

where

$$\begin{aligned} \lambda &= a\,i\ +b\,j\ +c\,k \\ \mu &= a'i\ +b'j\ +c'k \\ \nu &= a''i+b''j+c''k. \end{aligned}$$

119. By a similar process, the sum of three dyads may be reduced to the sum of two dyads, whenever either the antecedents or the consequents are complanar, and only in such cases. To prove the latter point, let us suppose that in the dyadic

$$\alpha\lambda+\beta\mu+\gamma\nu$$

neither the antecedents nor the consequents are complanar. The vector

$$\{\alpha\lambda+\beta\mu+\gamma\nu\}.\rho$$

is a linear function of ρ which will be parallel to α when ρ is perpendicular to μ and ν, which will be parallel to β when ρ is perpendicular to ν and λ, and which will be parallel to γ when ρ is perpendicular to λ and μ. Hence, the function may be given any value whatever by giving the proper value to ρ. This would evidently not be the case with the sum of two dyads. Hence, by No. 108, this dyadic cannot be equal to the sum of two dyads.

120. In like manner, the sum of two dyads may be reduced to a single dyad, if either the antecedents or the consequents are parallel, and only in such cases.

A sum of three dyads cannot be reduced to a single dyad, unless either their antecedents or consequents are parallel, or both antecedents and consequents are (separately) complanar. In the first case the reduction can always be made, in the second, occasionally.

121. *Def.*—A dyadic which cannot be reduced to the sum of less than three dyads will be called *complete.*

A dyadic which can be reduced to the sum of two dyads will be called *planar.* When the plane of the antecedents coincides with that of the consequents, the dyadic will be called *uniplanar.* These planes are invariable for a given dyadic, although the dyadic may be so expressed that either the two antecedents or the two consequents may have any desired values (which are not parallel) within their planes.

A dyadic which can be reduced to a single dyad will be called *linear.* When the antecedent and consequent are parallel, it will be called *unilinear.*

A dyadic is said to have the value zero when all its terms vanish.

122. If we set

$$\sigma = \Phi.\rho, \qquad \tau = \rho.\Phi,$$

and give ρ all possible values, σ and τ will receive all possible values, if Φ is complete. The values of σ and τ will be confined each to a plane if Φ is planar, which planes will coincide if Φ is uniplanar. The values of σ and τ will be confined each to a line if Φ is linear, which lines will coincide if Φ is unilinear.

123. The products of complete dyadics are complete, of complete and planar dyadics are planar, of complete and linear dyadics are linear.

The products of planar dyadics are planar, except that when the plane of the consequents of the first dyadic is perpendicular to the plane of the antecedents of the second dyadic, the product reduces to a linear dyadic.

The products of linear dyadics are linear, except that when the consequent of the first is perpendicular to the antecedent of the second, the product reduces to zero.

The products of planar and linear dyadics are linear, except when, the planar preceding, the plane of its consequents is perpendicular to the antecedent of the linear, or, the linear preceding, its consequent is perpendicular to the plane of the antecedents of the planar. In these cases the product is zero.

All these cases are readily proved, if we set

$$\sigma = \Phi.\Psi.\rho,$$

and consider the limits within which σ varies, when we give ρ all possible values.

The products $\Psi\times\rho$ and $\rho\times\Phi$ are evidently planar dyadics.

124. *Def.*—A dyadic Φ is said to be an *idemfactor*, when

$$\Phi.\rho=\rho \text{ for all values of } \rho,$$

or when

$$\rho.\Phi=\rho \text{ for all values of } \rho.$$

If either of these conditions holds true, Φ must be reducible to the form

$$ii+jj+kk.$$

Therefore, both conditions will hold, if either does. All such dyadics are equal, by No. 108. They will be represented by the letter I.

The direct product of an idemfactor with another dyadic is equal to that dyadic. That is,

$$\mathrm{I}.\Phi=\Phi, \qquad \Phi.\mathrm{I}=\Phi,$$

where Φ is any dyadic.

A dyadic of the form

$$\alpha\alpha'+\beta\beta'+\gamma\gamma',$$

in which α', β', γ' are the reciprocals of α, β, γ, is an idemfactor. (See No. 38.) A dyadic trinomial cannot be an idemfactor, unless its antecedents and consequents are reciprocals.

125. If one of the direct products of two dyadics is an idemfactor, the other is also. For, if $\Phi.\Psi=\mathrm{I}$,

$$\sigma.\Phi.\Psi=\sigma$$

for all values of σ, and Φ is complete;

$$\sigma.\Phi.\Psi.\Phi=\sigma.\Phi$$

for all values of σ, therefore for all values of $\sigma.\Phi$, and therefore $\Psi.\Phi=\mathrm{I}$.

Def.—In this case, either dyadic is called the *reciprocal* of the other.

It is evident that an incomplete dyadic cannot have any (finite) reciprocal.

Reciprocals of the same dyadic are equal. For if Φ and Ψ are both reciprocals of Ω,

$$\Phi=\Phi.\Omega.\Psi=\Psi.$$

If two dyadics are reciprocals, the operators formed by using these dyadics as prefactors are inverse, also the operators formed by using them as postfactors.

126. The reciprocal of any complete dyadic

$$\alpha\lambda+\beta\mu+\gamma\nu$$

is

$$\lambda'\alpha'+\mu'\beta'+\nu'\gamma',$$

where α', β', γ' are the reciprocals of α, β, γ, and λ', μ', ν' are the reciprocals of λ, μ, ν. (See No. 38.)

127. *Def.*—We shall write Φ^{-1} for the reciprocal of any (complete) dyadic Φ, also Φ^2 for $\Phi.\Phi$, etc., and Φ^{-2}, for $\Phi^{-1}.\Phi^{-1}$, etc. It is evident that Φ^{-n} is the reciprocal of Φ^n,

128. In the reduction of equations, if we have

$$\Phi.\Psi = \Phi.\Omega,$$

we may cancel the Φ (which is equivalent to multiplying by Φ^{-1}) if Φ is a complete dyadic, but not otherwise. The case is the same with such equations as

$$\Phi.\sigma = \Phi.\rho, \quad \Psi.\Phi = \Omega.\Phi, \quad \rho.\Phi = \sigma.\Phi.$$

To cancel an incomplete dyadic in such cases would be analogous to cancelling a zero factor in algebra.

129. *Def.*—If in any dyadic we transpose the factors in each term, the dyadic thus formed is said to be *conjugate* to the first. Thus

$$a\lambda + \beta\mu + \gamma\nu \text{ and } \lambda a + \mu\beta + \nu\gamma$$

are conjugate to each other. A dyadic of which the value is not altered by such transposition is said to be *self-conjugate*. The conjugate of any dyadic Φ may be written Φ_0. It is evident that

$$\rho.\Phi = \Phi_0.\rho \text{ and } \Phi.\rho = \rho.\Phi_0.$$

$\Phi_0.\rho$ and $\Phi.\rho$ are conjugate functions of ρ. (See No. 106.) Since $\{\Phi_0\}^2 = \{\Phi^2\}_0$, we may write Φ_0^2, etc., without ambiguity.

130. The reciprocal of the product of any number of dyadics is equal to the product of their reciprocals taken in inverse order. Thus

$$\{\Phi.\Psi.\Omega\}^{-1} = \Omega^{-1}.\Psi^{-1}.\Phi^{-1}.$$

The conjugate of the product of any number of dyadics is equal to the product of their conjugates taken in inverse order. Thus

$$\{\Phi.\Psi.\Omega\}_0 = \Omega_0.\Psi_0.\Phi_0.$$

Hence, since

$$\Phi_0.\{\Phi^{-1}\}_0 = \{\Phi^{-1}.\Phi\}_0 = I,$$
$$\{\Phi^{-1}\}_0 = \{\Phi_0\}^{-1},$$

and we may write Φ_0^{-1} without ambiguity.

131. It is sometimes convenient to be able to express by a dyadic taken in direct multiplication the same operation which would be effected by a given vector (a) in skew multiplication. The dyadic $I \times a$ will answer this purpose. For, by No. 117,

$$\{I \times a\}.\rho = a \times \rho, \qquad \rho.\{I \times a\} = \rho \times a,$$
$$\{I \times a\}.\Phi = a \times \Phi, \qquad \Phi.\{I \times a\} = \Phi \times a.$$

The same is true of the dyadic $a \times I$, which is indeed identical with $I \times a$, as appears from the equation $I.\{a \times I\} = \{I \times a\}.I$.

If α is a unit vector,

$$\{\mathrm{I}\times\alpha\}^2=-\{\mathrm{I}-\alpha\alpha\},$$
$$\{\mathrm{I}\times\alpha\}^3=-\mathrm{I}\times\alpha,$$
$$\{\mathrm{I}\times\alpha\}^4=\mathrm{I}-\alpha\alpha,$$
$$\{\mathrm{I}\times\alpha\}^5=\mathrm{I}\times\alpha,$$
$$\text{etc.}$$

If i, j, k are a normal system of unit vectors

$$\mathrm{I}\times i=i\times\mathrm{I}=kj-jk.$$
$$\mathrm{I}\times j=j\times\mathrm{I}=ik-ki,$$
$$\mathrm{I}\times k=k\times\mathrm{I}=ji-ij.$$

If α and β are any vectors,

$$[\alpha\times\beta]\times\mathrm{I}=\mathrm{I}\times[\alpha\times\beta]=\beta\alpha-\alpha\beta.$$

That is, the vector $\alpha\times\beta$ as a pre- or post-factor in skew multiplication is equivalent to the dyadic $\{\beta\alpha-\alpha\beta\}$ taken as pre- or post-factor in direct multiplication.

$$[\alpha\times\beta]\times\rho=\{\beta\alpha-\alpha\beta\}.\rho,$$
$$\rho\times[\alpha\times\beta]=\rho.\{\beta\alpha-\alpha\beta\}.$$

This is essentially the theorem of No. 27, expressed in a form more symmetrical, and more easily remembered.

132. The equation

$$\alpha\,\beta\times\gamma+\beta\,\gamma\times\alpha+\gamma\,\alpha\times\beta=\alpha.\beta\times\gamma\,\mathrm{I},$$

gives, on multiplication by any vector ρ, the identical equation

$$\rho.\alpha\,\beta\times\gamma+\rho.\beta\,\gamma\times\alpha+\rho.\gamma\,\alpha\times\beta=\alpha.\beta\times\gamma\,\rho.$$

(See No. 37.) The former equation is therefore identically true. (See No. 108.) It is a little more general than the equation

$$\alpha\alpha'+\beta\beta'+\gamma\gamma'=\mathrm{I},$$

which we have already considered (No. 124), since, in the form here given, it is not necessary that α, β, and γ should be non-complanar. We may also write

$$\beta\times\gamma\,\alpha+\gamma\times\alpha\,\beta+\alpha\times\beta\,\gamma=\alpha.\beta\times\gamma\,\mathrm{I}.$$

Multiplying this equation by ρ as prefactor (or the first equation by ρ as postfactor), we obtain

$$\rho.\beta\times\gamma\,\alpha+\rho.\gamma\times\alpha\,\beta+\rho.\alpha\times\beta\,\gamma=\alpha.\beta\times\gamma\,\rho.$$

(Compare No. 37.) For three complanar vectors we have

$$\alpha\,\beta\times\gamma+\beta\,\gamma\times\alpha+\gamma\,\alpha\times\beta=0.$$

Multiplying this by ν, a unit normal to the plane of α, β, and γ we have

$$\alpha\,\beta\times\gamma.\nu+\beta\,\gamma\times\alpha.\nu+\gamma\,\alpha\times\beta.\nu=0.$$

This equation expresses the well-known theorem that if the geometrical sum of three vectors is zero, the magnitude of each vector is proportional to the sine of the angle between the other two. It also indicates the numerical coefficients by which one of three complanar vectors may be expressed in parts of the other two.

133. *Def.*—If two dyadics Φ and Ψ are such that

$$\Phi.\Psi = \Psi.\Phi,$$

they are said to be *homologous.*

If any number of dyadics are homologous to one another, and any other dyadics are formed from them by the operations of taking multiples, sums, differences, powers, reciprocals, or products, such dyadics will be homologous to each other and to the original dyadics. This requires demonstration only in regard to reciprocals. Now if

$$\Phi.\Psi = \Psi.\Phi, \cdot$$

$$\Psi.\Phi^{-1} = \Phi^{-1}.\Phi.\Psi.\Phi^{-1} = \Phi^{-1}.\Psi.\Phi.\Phi^{-1} = \Phi^{-1}.\Psi.$$

That is, Φ^{-1} is homologous to Ψ, if Φ is.

134. If we call $\Psi.\Phi^{-1}$ or $\Phi^{-1}.\Psi$ the quotient of Ψ and Φ, we may say that the rules of addition, subtraction, multiplication and division of homologous dyadics are identical with those of arithmetic or ordinary algebra, except that limitations analogous to those respecting zero in algebra must be observed with respect to all incomplete dyadics.

It follows that the algebraic and higher analysis of homologous dyadics is substantially identical with that of scalars.

135. It is always possible to express a dyadic in three terms, so that both the antecedents and the consequents shall be perpendicular among themselves.

To show this for any dyadic Φ, let us set

$$\rho' = \Phi.\rho,$$

ρ being a unit-vector, and consider the different values of ρ' for all possible directions of ρ. Let the direction of the unit vector i be so determined that when ρ coincides with i, the value of ρ' shall be at least as great as for any other direction of ρ. And let the direction of the unit vector j be so determined that when ρ coincides with j, the value of ρ' shall be at least as great as for any other direction of ρ which is perpendicular to i. Let k have its usual position with respect to i and j. It is evidently possible to express Φ in the form

$$\alpha i + \beta j + \gamma k.$$

We have therefore

$$\rho' = \{\alpha i + \beta j + \gamma k\}.\rho,$$

and

$$d\rho' = \{\alpha i + \beta j + \gamma k\}.d\rho.$$

Now the supposed property of the direction of i requires that when ρ coincides with i and $d\rho$ is perpendicular to i, $d\rho'$ shall be perpendicular to ρ', which will then be parallel to α. But if $d\rho$ is parallel to j or k, it will be perpendicular to i, and $d\rho'$ will be parallel to β or γ, as the case may be. Therefore β and γ are perpendicular to α. In the same way it may be shown that the condition relative to j requires that γ shall be perpendicular to β. We may therefore set

$$\Phi = ai'i + bj'j + ck'k,$$

where i', j', k', like i, j, k, constitute a normal system of unit vectors (see No. 11), and a, b, c are scalars which may be either positive or negative.

It makes an important difference whether the number of these scalars which are negative is even or odd. If two are negative, say a and b, we may make them positive by reversing the directions of i' and j'. The vectors i', j', k' will still constitute a normal system. But if we should reverse the directions of an odd number of these vectors, they would cease to constitute a normal system, and to be superposable upon the system i, j, k. We may, however, always set either

$$\Phi = ai'i + bj'j + ck'k,$$

or

$$\Phi = -\{ai'i + bj'j + ck'k\},$$

with positive values of a, b, and c. At the limit between these cases are the planar dyadics, in which one of the three terms vanishes, and the dyadic reduces to the form

$$ai'i + bj'j,$$

in which a and b may always be made positive by giving the proper directions to i' and j'.

If the numerical values of a, b, c are all unequal, there will be only one way in which the value of Φ may be thus expressed. If they are not all unequal, there will be an infinite number of ways in which Φ may be thus expressed, in all of which the three scalar coefficients will have the same values with exception of the changes of signs mentioned above. If the three values are numerically identical, we may give to either system of normal vectors an arbitrary position.

136. It follows that any self-conjugate dyadic may be expressed in the form

$$aii + bjj + ckk,$$

where i, j, k are a normal system of unit vectors, and a, b, c are positive or negative scalars.

137. Any dyadic may be divided into two parts, of which one shall be self-conjugate, and the other of the form $\mathrm{I}\times\alpha$. These parts are found by taking half the sum and half the difference of the dyadic and its conjugate. It is evident that

$$\Phi = \tfrac{1}{2}\{\Phi + \Phi_0\} + \tfrac{1}{2}\{\Phi - \Phi_0\}.$$

Now $\frac{1}{2}\{\Phi+\Phi_0\}$ is self-conjugate, and

$$\tfrac{1}{2}\{\Phi-\Phi_0\}=\mathrm{I}\times[-\tfrac{1}{2}\Phi_\times].$$

(See No. 131.)

Rotations and Strains.

138. To illustrate the use of dyadics as operators, let us suppose that a body receives such a displacement that

$$\rho'=\Phi.\rho,$$

ρ and ρ' being the position-vectors of the same point of the body in its initial and subsequent positions. The same relation will hold of the vectors which unite any two points of the body in their initial and subsequent positions. For if ρ_1, ρ_2 are the original position-vectors of the points, and ρ_1', ρ_2' their final position-vectors, we have

$$\rho_1'=\Phi.\rho_1, \qquad \rho_2'=\Phi.\rho_2,$$

whence

$$\rho_2'-\rho_1'=\Phi.[\rho_2-\rho_1].$$

In the most general case, the body is said to receive a *homogeneous strain.* In special cases, the displacement reduces to a rotation. Lines in the body initially straight and parallel will be straight and parallel after the displacement, and surfaces initially plane and parallel will be plane and parallel after the displacement.

139. The vectors (σ, σ') which represent any plane surface in the body in its initial and final positions will be linear functions of each other. (This will appear, if we consider the four sides of a tetrahedron in the body.) To find the relation of the dyadics which express σ' as a function of σ, and ρ' as a function of ρ, let

$$\rho'=\{\alpha\lambda+\beta\mu+\gamma\nu\}.\rho.$$

Then, if we write λ', μ', ν' for the reciprocals of λ, μ, ν, the vectors λ', μ', ν' become by the strain α, β, γ. Therefore the surfaces $\mu'\times\nu'$, $\nu'\times\lambda'$, $\lambda'\times\mu'$ become $\beta\times\gamma$, $\gamma\times\alpha$, $\alpha\times\beta$. But $\mu'\times\nu'$, $\nu'\times\lambda'$, $\lambda'\times\mu'$ are the reciprocals of $\mu\times\nu$, $\nu\times\lambda$, $\lambda\times\mu$. The relation sought is therefore

$$\sigma'=\{\beta\times\gamma\,\mu\times\nu+\gamma\times\alpha\,\nu\times\lambda+\alpha\times\beta\,\lambda\times\mu\}.\sigma.$$

140. The volume $\lambda'.\mu'\times\nu'$ becomes by the strain $\alpha.\beta\times\gamma$. The unit of volume becomes therefore $(\alpha.\beta\times\gamma)(\lambda.\mu\times\nu)$.

Def.—It follows that the scalar product of the three antecedents multiplied by the scalar product of the three consequents of a dyadic expressed as a trinomial is independent of the particular form in which the dyadic is thus expressed. This quantity is the determinant of the coefficients of the nine terms of the form

$$aii+bij+\text{etc.},$$

into which the dyadic may be expanded. We shall call it the *determinant* of the dyadic, and shall denote it by the notation

$$|\Phi|$$

when the dyadic is expressed by a single letter.

If a dyadic is incomplete, its determinant is zero, and conversely.

The determinant of the product of any number of dyadics is equal to the product of their determinants. The determinant of the reciprocal of a dyadic is the reciprocal of the determinant of that dyadic. The determinants of a dyadic and its conjugate are equal.

The relation of the surfaces σ' and σ may be expressed by the equation

$$\sigma' = |\Phi| \Phi_0^{-1}.\sigma.*$$

141. Let us now consider the different cases of rotation and strain as determined by the nature of the dyadic Φ.

If Φ is reducible to the form

$$i'i+j'j+k'k,$$

i, j, k, i', j', k' being normal systems of unit vectors (see No. 11), the body will suffer no change of form. For if

$$\rho = xi+yj+zk,$$

we shall have

$$\rho' = xi'+yj'+zk'.$$

Conversely, if the body suffers no change of form, the operating dyadic is reducible to the above form. In such cases, it appears from simple geometrical considerations that the displacement of the body may be produced by a rotation about a certain axis. A dyadic reducible to the form

$$i'i+j'j+k'k$$

may therefore be called a *versor*.

142. The conjugate operator evidently produces the reverse rotation. A versor, therefore, is the reciprocal of its conjugate.

Conversely, if a dyadic is the reciprocal of its conjugate, it is either a versor, or a versor multiplied by -1. For the dyadic may be expressed in the form

$$\alpha i+\beta j+\gamma k.$$

Its conjugate will be $\quad i\alpha+j\beta+k\gamma.$

If these are reciprocals, we have

$$\{\alpha i+\beta j+\gamma k\}.\{i\alpha+j\beta+k\gamma\} = \alpha\alpha+\beta\beta+\gamma\gamma = \mathrm{I}.$$

But this relation cannot subsist unless α, β, γ are reciprocals to themselves, i.e., unless they are mutually perpendicular unit-vectors. Therefore, they either are a normal system of unit-vectors, or will become such if their directions are reversed. Therefore, one of the dyadics

$$\alpha i+\beta j+\gamma k \quad \text{and} \quad -\alpha i-\beta j-\gamma k$$

is a versor.

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*[See note on p. 90.]

The criterion of a versor may therefore be written

$$\Phi.\Phi_0=\mathrm{I}, \quad \text{and} \quad |\Phi|=1.$$

For the last equation we may substitute

$$|\Phi|>0, \quad \text{or} \quad |\Phi|\gtrless -1.$$

It is evident that the resultant of successive finite rotations is obtained by multiplication of the versors.

143. If we take the axis of the rotation for the direction of i, i' will have the same direction, and the versor reduces to the form

$$ii+j'j+k'k,$$

in which i, j, k and i, j', k' are normal systems of unit vectors.

We may set

$$j'=\cos q\, j + \sin q\, k,$$

$$k'=\cos q\, k - \sin q\, j,$$

and the versor reduces to

$$ii + \cos q\,\{jj+kk\} + \sin q\,\{kj-jk\},$$

or

$$ii + \cos q\,\{\mathrm{I}-ii\} + \sin q\,\mathrm{I}\times i,$$

where q is the angle of rotation, measured from j toward k, if the versor is used as a prefactor.

144. When any versor Φ is used as a prefactor, the vector $-\Phi_\times$ will be parallel to the axis of rotation, and equal in magnitude to twice the sine of the angle of rotation measured counter-clockwise as seen from the direction in which the vector points. (This will appear if we suppose Φ to be represented in the form given in the last paragraph.) The scalar Φ_s will be equal to unity increased by twice the cosine of the same angle. Together, $-\Phi_\times$ and Φ_s determine the versor without ambiguity. If we set

$$\theta=\frac{-\Phi_\times}{1+\Phi_s},$$

the magnitude of θ will be

$$\frac{2\sin q}{2+2\cos q} \quad \text{or} \quad \tan \tfrac{1}{2}q,$$

where q is measured counter-clockwise as seen from the direction in which θ points. This vector θ, which we may call the *vector semi-tangent of version*, determines the versor without ambiguity.

145. The versor Φ may be expressed in terms of θ in various ways. Since Φ (as prefactor) changes $a-\theta\times a$ into $a+\theta\times a$ (a being any vector), we have

$$\Phi=\{\mathrm{I}+\mathrm{I}\times\theta\}.\{\mathrm{I}-\mathrm{I}\times\theta\}^{-1}.$$

Again

$$\Phi=\frac{\theta\theta+\{\mathrm{I}+\mathrm{I}\times\theta\}^2}{1+\theta.\theta}=\frac{(1-\theta.\theta)\mathrm{I}+2\theta\theta+2\mathrm{I}\times\theta}{1+\theta.\theta},$$

as will be evident on considering separately in the expression $\Phi.\rho$ the components perpendicular and parallel to θ, or on substituting in

$$ii + \cos q\,(jj+kk) + \sin q\,(kj-jk)$$

for $\cos q$ and $\sin q$ their values in terms of $\tan \frac{1}{2}q$.

If we set, in either of these equations,

$$\theta = ai+bj+ck,$$

we obtain, on reduction, the formula

$$\Phi = \frac{\left\{\begin{array}{l}(1+a^2-b^2-c^2)ii+(2ab-2c)ij+(2ac+2b)ik\\ +(2ab+2c)ji+(1-a^2+b^2-c^2)jj+(2bc-2a)jk\\ +(2ac-2b)ki+(2bc+2a)kj+(1-a^2-b^2+c^2)kk\end{array}\right\}}{1+a^2+b^2+c^2},$$

in which the versor is expressed in terms of the rectangular components of the vector semitangent of version.

146. If α, β, γ are unit vectors, expressions of the form

$$2\alpha\alpha-\mathrm{I}, \qquad 2\beta\beta-\mathrm{I}, \qquad 2\gamma\gamma-\mathrm{I},$$

are biquadrantal versors. A product like

$$\{2\beta\beta-\mathrm{I}\}.\{2\alpha\alpha-\mathrm{I}\}$$

is a versor of which the axis is perpendicular to α and β, and the amount of rotation twice that which would carry α to β. It is evident that any versor may be thus expressed, and that either α or β may be given any direction perpendicular to the axis of rotation. If

$$\Phi = \{2\beta\beta-\mathrm{I}\}.\{2\alpha\alpha-\mathrm{I}\}, \quad \text{and} \quad \Psi = \{2\gamma\gamma-\mathrm{I}\}.\{2\beta\beta-\mathrm{I}\},$$

we have for the resultant of the successive rotations

$$\Psi.\Phi = \{2\gamma\gamma-\mathrm{I}\}.\{2\alpha\alpha-\mathrm{I}\}.$$

This may be applied to the composition of any two successive rotations, β being taken perpendicular to the two axes of rotation, and affords the means of determining the resultant rotation by construction on the surface of a sphere. It also furnishes a simple method of finding the relations of the vector semitangents of version for the versors Φ, Ψ, and $\Psi.\Phi$. Let

$$\theta_1 = \frac{-\Phi_\times}{1+\Phi_s}, \qquad \theta_2 = \frac{-\Psi_\times}{1+\Psi_s}, \qquad \theta_3 = \frac{-\{\Psi.\Phi\}_\times}{1+\{\Psi.\Phi\}_s}.$$

Then, since

$$\Phi = 4\,\alpha.\beta\beta\alpha - 2\alpha\alpha - 2\beta\beta + \mathrm{I},$$

$$\theta_1 = \frac{\alpha\times\beta}{\alpha.\beta},$$

which is moreover geometrically evident. In like manner,

$$\theta_2 = \frac{\beta\times\gamma}{\beta.\gamma}, \qquad \theta_3 = \frac{\alpha\times\gamma}{\alpha.\gamma}.$$

Therefore,

$$\theta_1\times\theta_2=\frac{[\alpha\times\beta]\times[\beta\times\gamma]}{\alpha.\beta\,\beta.\gamma}=\frac{\alpha\times\beta.\gamma\,\beta}{\alpha.\beta\,\beta.\gamma}$$

$$=\frac{\beta.\alpha\,\beta\times\gamma+\beta.\beta\,\gamma\times\alpha+\beta.\gamma\,\alpha\times\beta}{\alpha.\beta\,\beta.\gamma}.$$

(See No. 38.) That is,

$$\theta_1\times\theta_2=\theta_2-\frac{\alpha.\gamma}{\alpha.\beta\,\beta.\gamma}\theta_3+\theta_1.$$

Also,
$$\theta_1.\theta_2=\frac{\alpha\times\beta.\beta\times\gamma}{\alpha.\beta\,\beta.\gamma}=1-\frac{\alpha.\gamma}{\alpha.\beta\,\beta.\gamma}.$$

Hence,
$$\theta_1\times\theta_2=\theta_2-(1-\theta_1.\theta_2)\theta_3+\theta_1,$$

$$\theta_3=\frac{\theta_1+\theta_2+\theta_2\times\theta_1}{1-\theta_1.\theta_2},$$

which is the formula for the composition of successive finite rotations by means of their vector semitangents of version.

147. The versors just described constitute a particular class under the more general form

$$\alpha\alpha'+\cos q\{\beta\beta'+\gamma\gamma'\}+\sin q\{\gamma\beta'-\beta\gamma'\},$$

in which α, β, γ are any non-complanar vectors, and α', β', γ' their reciprocals. A dyadic of this form as a prefactor does not affect any vector parallel to α. Its effect on a vector in the β-γ plane will be best understood if we imagine an ellipse to be described of which β and γ are conjugate semi-diameters. If the vector to be operated on be a radius of this ellipse, we may evidently regard the ellipse with β, γ, and the other vector, as the projections of a circle with two perpendicular radii and one other radius. A little consideration will show that if the third radius of the circle is advanced an angle q, its projection in the ellipse will be advanced as required by the dyadic prefactor. The effect, therefore, of such a prefactor on a vector in the β-γ plane may be obtained as follows: Describe an ellipse of which β and γ are conjugate semi-diameters. Then describe a similar and similarly placed ellipse of which the vector to be operated on is a radius. The effect of the operator is to advance the radius in this ellipse, in the angular direction from β toward γ, over a segment which is to the total area of the ellipse as q is to 2π. When used as a postfactor, the properties of the dyadic are similar, but the axis of no motion and the planes of rotation are in general different.

Def.—Such dyadics we shall call *cyclic.*

The Nth power (N being any whole number) of such a dyadic is obtained by multiplying q by N. If q is of the form $2\pi N/M$ (N and M being any whole numbers) the Mth power of the dyadic will be an idemfactor. A cyclic dyadic, therefore, may be regarded as a root of

I, or at least capable of expression with any required degree of accuracy as a root of I.

It should be observed that the value of the above dyadic will not be altered by the substitution for α of any other parallel vector, or for β and γ of any other conjugate semi-diameters (which succeed one another in the same angular direction) of the same or any similar and similarly situated ellipse, with the changes which these substitutions require in the values of α', β', γ'. Or, to consider the same changes from another point of view, the value of the dyadic will not be altered by the substitution for α' of any other parallel vector or for β' and γ' of any other conjugate semi-diameters (which succeed one another in the same angular direction) of the same or any similar and similarly situated ellipse, with the changes which these substitutions require in the values of α, β, and γ, defined as reciprocals of α', β', γ'.

148. The strain represented by the equation

$$\rho' = \{aii + bjj + ckk\} . \rho$$

where a, b, c are positive scalars, may be described as consisting of three elongations (or contractions) parallel to the axes i, j, k, which are called the *principal axes of the strain*, and which have the property that their directions are not affected by the strain. The scalars a, b, c are called the *principal ratios of elongation*. (When one of these is less than unity, it represents a contraction.) The order of the three elongations is immaterial, since the original dyadic is equal to the product of the three dyadics

$$aii + jj + kk, \qquad ii + bjj + kk, \qquad ii + jj + ckk$$

taken in any order.

Def.—A dyadic which is reducible to this form we shall call a *right tensor*. The displacement represented by a right tensor is called a *pure strain*. A right tensor is evidently self-conjugate.

149. We have seen (No. 135) that every dyadic may be expressed in the form

$$\pm \{ai'i + bj'j + ck'k\},$$

where a, b, c are positive scalars. This is equivalent to

$$\pm \{ai'i' + bj'j' + ck'k'\} . \{i'i + j'j + k'k\}$$

and to

$$\pm \{i'i + j'j + k'k\} . \{aii + bjj + ckk\}.$$

Hence every dyadic may be expressed as the product of a versor and a right tensor with the scalar factor ± 1. The versor may precede or follow. It will be the same versor in either case, and the ratios of elongation will be the same; but the position of the principal axes of the tensor will differ in the two cases, either system being derived from the other by multiplication by the versor.

Def.—The displacement represented by the equation

$$\rho' = -\rho$$

is called *inversion*. The most general case of a homogeneous strain may therefore be produced by a pure strain and a rotation with or without inversion.

150. If

$$\Phi = ai'i + bj'j + ck'k,$$

$$\Phi.\Phi_0 = a^2 i'i' + b^2 j'j' + c^2 k'k',$$

and

$$\Phi_0.\Phi = a^2 ii + b^2 jj + c^2 kk.$$

The general problem of the determination of the principal ratios and axes of strain for a given dyadic may thus be reduced to the case of a right tensor.

151. *Def.*—The effect of a prefactor of the form

$$a\alpha\alpha' + b\beta\beta' + c\gamma\gamma',$$

where a, b, c are positive or negative scalars, α, β, γ non-complanar vectors, and α', β', γ' their reciprocals, is to change α into $a\alpha$, β into $b\beta$, and γ into $c\gamma$. As a postfactor, the same dyadic will change α' into $a\alpha'$, β' into $b\beta'$, and γ' into $c\gamma'$. Dyadics which can be reduced to this form we shall call *tonic* (Gr. $\tau\epsilon\acute{\iota}\nu\omega$). The right tensor already described constitutes a particular case, distinguished by perpendicular axes and positive values of the coefficients a, b, c.

The value of the dyadic is evidently not affected by substituting vectors of different lengths but the same or opposite directions for α, β, γ, with the necessary changes in the values of α', β', γ', defined as reciprocals of α, β, γ. But, except this change, if a, b, c are unequal, the dyadic can be expressed only in one way in the above form. If, however, two of these coefficients are equal, say a and b, any two non-collinear vectors in the α-β plane may be substituted for α and β, or, if the three coefficients are equal, any three non-complanar vectors may be substituted for α, β, γ.

152. Tonics having the same axes (determined by the directions of α, β, γ) are homologous, and their multiplication is effected by multiplying their coefficients. Thus,

$$\{a_1\alpha\alpha' + b_1\beta\beta' + c_1\gamma\gamma'\}.\{a_2\alpha\alpha' + b_2\beta\beta' + c_2\gamma\gamma'\}$$
$$= \{a_1a_2\alpha\alpha' + b_1b_2\beta\beta' + c_1c_2\gamma\gamma'\}.$$

Hence, division of such dyadics is effected by division of their coefficients. A tonic of which the three coefficients a, b, c are unequal, is homologous only with such dyadics as can be obtained by varying the coefficients.

153. The effect of a prefactor of the form

$$a\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\} + c\{\gamma\beta' - \beta\gamma'\},$$

or

$$a\alpha\alpha' + p\cos q\{\beta\beta' + \gamma\gamma'\} + p\sin q\{\gamma\beta' - \beta\gamma'\},$$

where α', β', γ' are the reciprocals of α, β, γ, and a, b, c, p, and q are scalars, of which p is positive, will be most evident if we resolve it into the factors

$$a\alpha\alpha'+\beta\beta'+\gamma\gamma',$$

$$a\alpha'+p\beta\beta'+p\gamma\gamma',$$

$$a\alpha'+\cos q\{\beta\beta'+\gamma\gamma'\}+\sin q\{\gamma\beta'-\beta\gamma'\},$$

of which the order is immaterial, and if we suppose the vector on which we operate to be resolved into two factors, one parallel to α, and the other in the β-γ plane. The effect of the first factor is to multiply by a the component parallel to α, without affecting the other. The effect of the second is to multiply by p the component in the β-γ plane without affecting the other. The effect of the third is to give the component in the β-γ plane the kind of elliptic rotation described in No. 147.

The effect of the same dyadic as a postfactor is of the same nature.

The value of the dyadic is not affected by the substitution for α of another vector having the same direction, nor by the substitution for β and γ of two other conjugate semi-diameters of the same or a similar and similarly situated ellipse, and which follow one another in the same angular direction.

Def.—Such dyadics we shall call *cyclotonic.*

154. Cyclotonics which are reducible to the same form except with respect to the values of α, p, and q are homologous. They are multiplied by multiplying the values of α, and also those of p, and adding those of q. Thus, the product of

$$a_1\alpha\alpha'+p_1\cos q_1\{\beta\beta'+\gamma\gamma'\}+p_1\sin q_1\{\gamma\beta'-\beta\gamma'\}$$

and

$$a_2\alpha\alpha'+p_2\cos q_2\{\beta\beta'+\gamma\gamma'\}+p_2\sin q_2\{\gamma\beta'-\beta\gamma'\}$$

is

$$a_1a_2\alpha\alpha'+p_1p_2\cos(q_1+q_2)\{\beta\beta'+\gamma\gamma'\}+p_1p_2\sin(q_1+q_2)\{\gamma\beta'-\beta\gamma'\}.$$

A dyadic of this form, in which the value of q is not zero, or the product of π and a positive or negative integer, is homologous only with such dyadics as are obtained by varying the values of α, p, and q.

155. In general, any dyadic may be reduced to the form either of a tonic or of a cyclotonic. (The exceptions are such as are made by the limiting cases.) We may show this, and also indicate how the reduction may be made, as follows. Let Φ be any dyadic. We have first to show that there is at least one direction of ρ for which

$$\Phi.\rho=a\rho.$$

This equation is equivalent to

$$\Phi.\rho-a\rho=0,$$

or,

$$\{\Phi-a\mathrm{I}\}.\rho=0.$$

That is, $\Phi - a\mathrm{I}$ is a planar dyadic, which may be expressed by the equation

$$|\Phi - a\mathrm{I}| = 0.$$

(See No. 140.) Let

$$\Phi = \lambda i + \mu j + \nu k;$$

the equation becomes

$$|[\lambda - ai]i + [\mu - aj]j + [\nu - ak]k| = 0,$$

or,

$$[\lambda - ai] \times [\mu - aj] . [\nu - ak] = 0,$$

or, $a^3 - (i.\lambda + j.\mu + k.\nu)a^2 + (i.\mu \times \nu + j.\nu \times \lambda + k.\lambda \times \mu)a - \lambda \times \mu . \nu = 0.$

This may be written

$$a^3 - \Phi_s a^2 + \{\Phi^{-1}\}_s |\Phi| a - |\Phi| = 0.*$$

Now if the dyadic Φ is given in any form, the scalars

$$\Phi_s, \quad \{\Phi^{-1}\}_s, \quad |\Phi|$$

are easily determined. We have therefore a cubic equation in a, for which we can find at least one and perhaps three roots. That is, we can find at least one value of a, and perhaps three, which will satisfy the equation

$$|\Phi - a\mathrm{I}| = 0.$$

By substitution of such a value, $\Phi - a\mathrm{I}$ becomes a planar dyadic, the planes of which may be easily determined.† Let α be a vector normal to the plane of the consequents. Then

$$\{\Phi - a\mathrm{I}\} . \alpha = 0,$$

$$\Phi . \alpha = a\alpha.$$

If Φ is a tonic, we may obtain three equations of this kind, say

$$\Phi . \alpha = a\alpha, \qquad \Phi . \beta = b\beta, \qquad \Phi . \gamma = c\gamma,$$

in which α, β, γ are not complanar. Hence (by No. 108),

$$\Phi = a\alpha\alpha' + b\beta\beta' + c\gamma\gamma',$$

where α', β', γ' are the reciprocals of α, β, γ.

In any case, we may suppose a to have the same sign as $|\Phi|$, since the cubic equation must have such a root. Let a (as before) be normal to the plane of the consequents of the planar $\Phi - a\mathrm{I}$, and α' normal to the plane of the antecedents, the lengths of α and α' being such that $\alpha . \alpha' = 1$.‡ Let β be any vector normal to α', and such that $\Phi . \beta$ is not parallel to β. (The case in which $\Phi . \beta$ is always parallel to β, if β is perpendicular to α', is evidently that of a tonic, and needs no farther discussion.) $\{\Phi - a\mathrm{I}\} . \beta$ and therefore $\Phi . \beta$ will be perpendicular to α'. The same will be true of $\Phi^2 . \beta$. Now (by No. 140)

$$[\Phi . \alpha] . [\Phi^2 . \beta] \times [\Phi . \beta] = |\Phi| \alpha . [\Phi . \beta] \times \beta,$$

that is,

$$a\alpha . [\Phi^2 . \beta] \times [\Phi . \beta] = |\Phi| \alpha . [\Phi . \beta] \times \beta.$$

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*[See note on p. 90.]

†In particular cases, $\Phi - a\mathrm{I}$ may reduce to a linear dyadic, or to zero. These, however, will present no difficulties to the student.

‡For the case in which the two planes are perpendicular to each other, see No. 157.

Hence, since $[\Phi^2.\beta]\times[\Phi.\beta]$ and $[\Phi.\beta]\times\beta$ are parallel,

$$\alpha[\Phi^2.\beta]\times[\Phi.\beta]=|\Phi|\,[\Phi.\beta]\times\beta.$$

Since $\alpha^{-1}|\Phi|$ is positive, we may set

$$p^2=\alpha^{-1}|\Phi|.$$

If we also set

$$\beta_1=p^{-1}\Phi.\beta, \qquad \beta_2=p^{-2}\Phi^2.\beta, \quad \text{etc.},$$
$$\beta_{-1}=p\Phi^{-1}.\beta, \qquad \beta_{-2}=p^2\Phi^{-2}.\beta, \quad \text{etc.},$$

the vectors β, β_1, β_2, etc., β_{-1}, β_{-2}, etc., will all lie in the plane perpendicular to α', and we shall have

$$\beta_2\times\beta_1=\beta_1\times\beta,$$
$$[\beta_2+\beta]\times\beta_1=0.$$

We may therefore set $\beta_2+\beta=2n\beta_1$.

Multiplying by $p^{-1}\Phi$, and by $p\Phi^{-1}$,

$$\beta_3+\beta_1=2n\beta_2, \qquad \beta_4+\beta_2=2n\beta_3, \quad \text{etc.},$$
$$\beta_1+\beta_{-1}=2n\beta, \qquad \beta+\beta_{-2}=2n\beta_{-1}, \quad \text{etc.}$$

Now, if $n>1$, and we lay off from a common origin the vectors

$$\beta,\ \beta_1,\ \beta_2,\ \text{etc.},\quad \beta_{-1},\ \beta_{-2},\ \text{etc.},$$

the broken line joining the termini of these vectors will be convex toward the origin. All these vectors must therefore lie between two limiting lines, which may be drawn from the origin, and which may be described as having the directions of β_∞ and $\beta_{-\infty}$.* A vector having either of these directions is unaffected in direction by multiplication by Φ. In this case, therefore, Φ is a tonic. If $n<-1$ we may obtain the same result by considering the vectors

$$\beta,\ -\beta_1,\ \beta_2,\ -\beta_3,\ \beta_4,\ \text{etc.},\ -\beta_{-1},\ \beta_{-2},\ -\beta_{-3},\ \text{etc.},$$

except that a vector in the limiting directions will be reversed in direction by multiplication by Φ, which implies that the two corresponding coefficients of the tonic are negative.

If $1>n>-1$,† we may set

$$n=\cos q.$$

Then $\beta_{-1}+\beta_1=2\cos q\,\beta.$

Let us now determine γ by the equation

$$\beta_1=\cos q\,\beta+\sin q\,\gamma.$$

This gives $\beta_{-1}=\cos q\,\beta-\sin q\,\gamma.$

Now α' is one of the reciprocals of α, β, and γ. Let β' and γ' be the others. If we set

$$\Psi=\cos q\{\beta\beta'+\gamma\gamma'\}+\sin q\{\gamma\beta'-\beta\gamma'\},$$

we have $\Psi.\alpha=0, \qquad \Psi.\beta=\beta_1, \qquad \Psi.\beta_{-1}=\beta.$

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* The termini of the vectors will in fact lie on a hyperbola,

† For the limiting cases, in which $n=1$, or $n=-1$, see No. 156.

Therefore, since
$$\{\alpha\alpha'+p\Psi\}.\alpha=\alpha\alpha=\Phi.\alpha,$$
$$\{\alpha\alpha'+p\Psi\}.\beta=p\beta_1=\Phi.\beta,$$
$$\{\alpha\alpha'+p\Psi\}.\beta_{-1}=p\beta=\Phi.\beta_{-1},$$
it follows (by No. 108) that

$$\Phi=\alpha\alpha'+p\Psi=\alpha\alpha'+p\cos q\{\beta\beta'+\gamma\gamma'\}+p\sin q\{\gamma\beta'-\beta\gamma'\}.$$

156. It will be sufficient to indicate (without demonstration) the forms of dyadics which belong to the particular cases which have been passed over in the preceding paragraph, so far as they present any notable peculiarities.

If $n=\pm 1$ (page 72), the dyadic may be reduced to the form

$$\alpha\alpha'+b\{\beta\beta'+\gamma\gamma'\}+bc\beta\gamma',$$

where α, β, γ are three non-complanar vectors, α', β', γ' their reciprocals, and a, b, c positive or negative scalars. The effect of this as an operator, will be evident if we resolve it into the three homologous factors

$$\alpha\alpha'+\beta\beta'+\gamma\gamma',$$
$$\alpha\alpha'+b\{\beta\beta'+\gamma\gamma'\},$$
$$\alpha\alpha'+\beta\beta'+\gamma\gamma'+c\beta\gamma'.$$

The displacement due to the last factor may be called a *simple shear*. It consists (when the dyadic is used as prefactor) of a motion parallel to β, and proportioned to the distance from the α-β plane. This factor may be called a *shearer*.

This dyadic is homologous with such as are obtained by varying the values of a, b, c, and only with such, when the values of a and b are different, and that of c other than zero.

157. If the planar $\Phi-a\mathrm{I}$ (page 71) has perpendicular planes, there may be another value of a, of the same sign as $|\Phi|$, which will give a planar which has not perpendicular planes. When this is not the case, the dyadic may always be reduced to the form

$$a\{\alpha\alpha'+\beta\beta'+\gamma\gamma'\}+ab\{\alpha\beta'+\beta\gamma'\}+ac\alpha\gamma',$$

where α, β, γ are three non-complanar vectors, α', β', γ', their reciprocals, and a, b, c, positive or negative scalars. This may be resolved into the homologous factors

$$a\mathrm{I} \text{ and } \mathrm{I}+b\{\alpha\beta'+\beta\gamma'\}+c\alpha\gamma'.$$

The displacement due to the last factor may be called a *complex shear*. It consists (when the dyadic is used as prefactor) of a motion parallel to α which is proportional to the distance from the α-γ plane, together with a motion parallel to $b\beta+c\alpha$ which is proportional to the distance from the α-β plane. This factor may be called a *complex shearer*

This dyadic is homologous with such as are obtained by varying the values of a, b, c, and only such, unless $b=0$.

It is always possible to take three mutually perpendicular vectors for α, β, and γ; or, if it be preferred, to take such values for these vectors as shall make the term containing c vanish.

158. The dyadics described in the two last paragraphs may be called *shearing dyadics.*

The criterion of a shearer is

$$\{\Phi-\mathrm{I}\}^3=0, \qquad \Phi-\mathrm{I}\neq 0.$$

The criterion of a simple shearer is

$$\{\Phi-\mathrm{I}\}^2=0, \qquad \Phi-\mathrm{I}\neq 0.$$

The criterion of a complex shearer is

$$\{\Phi-\mathrm{I}\}^3=0, \qquad \{\Phi-\mathrm{I}\}^2\neq 0.$$

NOTE.—If a dyadic Φ is a linear function of a vector ρ (the term *linear* being used in the same sense as in No. 105), we may represent the relation by an equation of the form

$$\Phi=a\beta\gamma.\rho+\epsilon\zeta\eta.\rho+\text{etc.},$$

or

$$\Phi=\{a\beta\gamma+\epsilon\zeta\eta+\text{etc.}\}.\rho,$$

where the expression in the braces may be called a *triadic polynomial*, and a single term $a\beta\gamma$ a *triad*, or the indeterminate product of the three vectors α, β, γ. We are thus led successively to the consideration of higher orders of indeterminate products of vectors, *triads*, *tetrads*, etc., in general *polyads*, and of polynomials consisting of such terms, *triadics*, *tetradics*, etc., in general *polyadics*. But the development of the subject in this direction lies beyond our present purpose.

It may sometimes be convenient to use notations like

$$\frac{\lambda,\ \mu,\ \nu}{|\alpha,\ \beta,\ \gamma} \text{ and } \frac{\lambda,\ \mu,\ \nu}{\alpha,\ \beta,\ \gamma|}$$

to represent the conjugate dyadics which, the first as prefactor, and the second as postfactor, change α, β, γ into λ, μ, ν, respectively. In the notations of the preceding chapter these would be written

$$/\lambda\alpha'+\mu\beta'+\nu\gamma' \text{ and } \alpha'\lambda+\beta'\mu+\gamma'\nu$$

respectively, α', β', γ' denoting the reciprocals of α, β, γ. If τ is a linear function of ρ, the dyadics which as prefactor and postfactor change ρ into τ may be written respectively

$$\frac{\tau}{|\rho} \text{ and } \frac{\tau}{\rho|}.$$

If τ is any function of ρ, the dyadics which as prefactor and postfactor change $d\rho$ into $d\tau$ may be written respectively

$$\frac{d\tau}{|d\rho} \text{ and } \frac{d\tau}{d\rho|}.$$

In the notation of the following chapter the second of these (when ρ denotes a position-vector) would be written $\nabla\tau$. The triadic which as prefactor changes $d\rho$ into $\frac{d\tau}{|d\rho}$ may be written $\frac{d^2\tau}{|d\rho^2}$, and that which as postfactor changes $d\rho$ into $\frac{d\tau}{d\rho|}$ may be written $\frac{d^2\tau}{d\rho^2|}$. The latter would be written $\nabla\nabla\tau$ in the notations of the following chapter.

CHAPTER IV.

(SUPPLEMENTARY TO CHAPTER II.)

CONCERNING THE DIFFERENTIAL AND INTEGRAL CALCULUS OF VECTORS.

159. If ω is a vector having continuously varying values in space, and ρ the vector determining the position of a point, we may set

$$\rho = xi + yj + zk,$$

$$d\rho = dx\,i + dy\,j + dz\,k,$$

and regard ω as a function of ρ, or of x, y, and z. Then,

$$d\omega = dx\frac{d\omega}{dx} + dy\frac{d\omega}{dy} + dz\frac{d\omega}{dz},$$

that is,
$$d\omega = d\rho \,.\left\{ i\frac{d\omega}{dx} + j\frac{d\omega}{dy} + k\frac{d\omega}{dz}\right\}.$$

If we set
$$\nabla\omega = i\frac{d\omega}{dx} + j\frac{d\omega}{dy} + k\frac{d\omega}{dz},$$

$$d\omega = d\rho \,.\nabla\omega.$$

Here ∇ stands for
$$i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$

exactly as in No. 52, except that it is here applied to a vector and produces a dyadic, while in the former case it was applied to a scalar and produced a vector. The dyadic $\nabla\omega$ represents the nine differential coefficients of the three components of ω with respect to x, y, and z, just as the vector ∇u (where u is a scalar function of ρ) represents the three differential coefficients of the scalar u with respect to x, y, and z.

It is evident that the expressions $\nabla.\omega$ and $\nabla\times\omega$ already defined (No. 54) are equivalent to $\{\nabla\omega\}_s$ and $\{\nabla\omega\}_\times$.

160. An important case is that in which the vector operated on is of the form ∇u. We have then

$$d\nabla u = d\rho \,.\nabla\nabla u,$$

where

$$\nabla\nabla u = \left\{ \begin{array}{lll} \dfrac{d^2u}{dx^2}ii & + \dfrac{d^2u}{dx\,dy}ij & + \dfrac{d^2u}{dx\,dz}ik \\ + \dfrac{d^2u}{dy\,dx}ji & + \dfrac{d^2u}{dy^2}jj & + \dfrac{d^2u}{dy\,dz}jk \\ + \dfrac{d^2u}{dz\,dx}ki & + \dfrac{d^2u}{dz\,dy}kj & + \dfrac{d^2u}{dz^2}kk. \end{array} \right\}$$

This dyadic, which is evidently self-conjugate, represents the six

differential coefficients of the second order of u with respect to x, y, and z.*

161. The operators $\nabla\times$ and $\nabla.$ may be applied to dyadics in a manner entirely analogous to their use with scalars. Thus we may define $\nabla\times\Phi$ and $\nabla.\Phi$ by the equations

$$\nabla\times\Phi=i\times\frac{d\Phi}{dx}+j\times\frac{d\Phi}{dy}+k\times\frac{d\Phi}{dz}$$

$$\nabla.\Phi=i.\frac{d\Phi}{dx}+j.\frac{d\Phi}{dy}+k.\frac{d\Phi}{dz}.$$

Then, if $$\Phi=ai+\beta j+\gamma k,$$

$$\nabla\times\Phi=\nabla\times ai+\nabla\times\beta j+\nabla\times\gamma k.$$

$$\nabla.\Phi=\nabla.ai+\nabla.\beta j+\nabla.\gamma k.$$

Or, if $$\Phi=ia+j\beta+k\gamma,$$

$$\nabla\times\Phi=i\left[\frac{d\gamma}{dy}-\frac{d\beta}{dz}\right]+j\left[\frac{da}{dz}-\frac{d\gamma}{dx}\right]+k\left[\frac{d\beta}{dx}-\frac{da}{dy}\right],$$

$$\nabla.\Phi=\frac{da}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}.$$

162. We may now regard $\nabla.\nabla$ in expressions like $\nabla.\nabla\omega$ as representing two successive operations, the result of which will be

$$\frac{d^2\omega}{dx^2}+\frac{d^2\omega}{dy^2}+\frac{d^2\omega}{dz^2}$$

in accordance with the definition of No. 70. We may also write $\nabla.\nabla\Phi$ for

$$\frac{d^2\Phi}{dx^2}+\frac{d^2\Phi}{dy^2}+\frac{d^2\Phi}{dz^2},$$

although in this case we cannot regard $\nabla.\nabla$ as representing two successive operations until we have defined $\nabla\Phi$.†

That $\nabla.\nabla\Phi=\nabla\nabla.\Phi-\nabla\times\nabla\times\Phi$ will be evident if we suppose Φ to be expressed in the form $ai+\beta j+\gamma k$. (See No. 71.)

163. We have already seen that

$$u''-u'=\int d\rho.\nabla u,$$

where u' and u'' denote the values of u at the beginning and the end of the line to which the integral relates. The same relation will hold for a vector; i.e.,

$$\omega''-\omega'=\int d\rho.\nabla\omega.$$

* We might proceed to higher steps in differentiation by means of the triadics $\nabla\nabla\omega$, $\nabla\nabla\nabla u$, the tetradics $\nabla\nabla\nabla\omega$, $\nabla\nabla\nabla\nabla u$, etc. See note on page 74. In like manner a dyadic function of position in space (Φ) might be differentiated by means of the triadic $\nabla\Phi$, the tetradic $\nabla\nabla\Phi$, etc.

† See footnote to No. 160.

164. The following equations between surface-integrals for a closed surface and volume-integrals for the space enclosed seem worthy of mention. One or two have already been given, and are here repeated for the sake of comparison.

$$\iint d\sigma\, u = \iiint dv\, \nabla u, \tag{1}$$

$$\iint d\sigma\, \omega = \iiint dv\, \nabla \omega, \tag{2}$$

$$\iint d\sigma . \omega = \iiint dv\, \nabla . \omega, \tag{3}$$

$$\iint d\sigma . \Phi = \iiint dv\, \nabla . \Phi, \tag{4}$$

$$\iint d\sigma \times \omega = \iiint dv\, \nabla \times \omega, \tag{5}$$

$$\iint d\sigma \times \Phi = \iiint dv\, \nabla \times \Phi. \tag{6}$$

It may aid the memory of the student to observe that the transformation may be effected in each case by substituting $\iiint dv\, \nabla$ for $\iint d\sigma$.

165. The following equations between line-integrals for a closed line and surface-integrals for any surface bounded by the line, may also be mentioned. (One of these has already been given. See No. 60.)

$$\int d\rho\, u = \iint d\sigma \times \nabla u, \tag{1}$$

$$\int d\rho\, \omega = \iint d\sigma \times \nabla \omega, \tag{2}$$

$$\int d\rho . \omega = \iint d\sigma . \nabla \times \omega, \tag{3}$$

$$\int d\rho . \Phi = \iint d\sigma . \nabla \times \Phi, \tag{4}$$

$$\int d\rho \times \omega = \iint \nabla \omega . d\sigma - \iint d\sigma \nabla . \omega. \tag{5}$$

These transformations may be effected by substituting $\iint [d\sigma \times \nabla]$ for $\int d\rho$. The brackets are here introduced to indicate that the multiplication of $d\sigma$ with the i, j, k implied in ∇ is to be performed before any other multiplication which may be required by a subsequent sign. (This notation is not recommended for ordinary use, but only suggested as a mnemonic artifice.)

166. To the equations in No. 65 may be added many others, as,

$$\nabla [u\omega] = \nabla u\, \omega + u \nabla \omega, \tag{1}$$

$$\nabla [\tau \times \omega] = \nabla \tau \times \omega - \nabla \omega \times \tau, \tag{2}$$

$$\nabla \times [\tau \times \omega] = \omega . \nabla \tau - \nabla . \tau \omega - \tau . \nabla \omega + \nabla . \omega \tau, \tag{3}$$

$$\nabla (\tau . \omega) = \nabla \tau . \omega + \nabla \omega . \tau, \tag{4}$$

$$\nabla . \{\tau \omega\} = \nabla . \tau \omega + \tau . \nabla \omega, \tag{5}$$

$$\nabla \times \{\tau \omega\} = \nabla \times \tau \omega - \tau \times \nabla \omega, \tag{6}$$

$$\nabla . \{u \Phi\} = \nabla u . \Phi + u \nabla . \Phi, \tag{7}$$

etc.

The principle in all these cases is that if we have one of the operators ∇, $\nabla .$, $\nabla \times$ prefixed to a product of any kind, and we make any

transformation of the expression which would be allowable if the ∇ were a vector (viz., by changes in the order of the factors, in the signs of multiplication, in the parentheses written or implied, etc.), by which changes the ∇ is brought into connection with one particular factor, the expression thus transformed will represent the part of the value of the original expression which results from the variation of that factor.

167. From the relations indicated in the last four paragraphs, may be obtained directly a great number of transformations of definite integrals similar to those given in Nos. 74-77, and corresponding to those known in the scalar calculus by the name of *integration by parts.*

168. The student will now find no difficulty in generalizing the integrations of differential equations given in Nos. 78-89 by applying to vectors those which relate to scalars, and to dyadics those which relate to vectors.

169. The propositions in No. 90 relating to minimum values of the volume-integral $\iiint u\omega.\omega\,dv$ may be generalized by substituting $\omega.\Phi.\omega$ for $u\omega.\omega$, Φ being a given dyadic function of position in space.

170. The theory of the integrals which have been called potentials, Newtonians, etc. (see Nos. 91-102) may be extended to cases in which the operand is a vector instead of a scalar or a dyadic instead of a vector. So far as the demonstrations are concerned, the case of a vector may be reduced to that of a scalar by considering separately its three components, and the case of a dyadic may be reduced to that of a vector, by supposing the dyadic expressed in the form $\phi i+\chi j+\omega k$ and considering each of these terms separately.

CHAPTER V.

CONCERNING TRANSCENDENTAL FUNCTIONS OF DYADICS.

171. *Def.*—The *exponential function*, the *sine* and the *cosine* of a dyadic may be defined by infinite series, exactly as the corresponding functions in scalar analysis, viz.,

$$e^{\Phi}=\mathrm{I}+\Phi+\tfrac{1}{2}\Phi^2+\tfrac{1}{2.3}\Phi^3+\text{etc.},$$
$$\sin\Phi=\Phi-\tfrac{1}{2.3}\Phi^3+\tfrac{1}{2.3.4.5}\Phi^5-\text{etc.},$$
$$\cos\Phi=\mathrm{I}-\tfrac{1}{2}\Phi^2+\tfrac{1}{2.3.4}\Phi^4-\text{etc.}$$

These series are always convergent. For every value of Φ there is one and only one value of each of these functions. The exponential function may also be defined as the limit of the expression

$$\left(\mathrm{I}+\frac{\Phi}{N}\right)^N,$$

when N, which is a whole number, is increased indefinitely. That this definition is equivalent to the preceding, will appear if the expression is expanded by the binomial theorem, which is evidently applicable in a case of this kind.

These functions of Φ are homologous with Φ.

172. We may define the logarithm as the function which is the inverse of the exponential, so that the equations

$$e^{\Psi} = \Phi,$$
$$\Psi = \log \Phi,$$

are equivalent, leaving it undetermined for the present whether every dyadic has a logarithm, and whether a dyadic can have more than one.

173. It follows at once from the second definition of the exponential function that, if Φ and Ψ are homologous,

$$e^{\Phi} . e^{\Psi} = e^{\Phi+\Psi},$$

and that, if T is a positive or negative whole number,

$$\{e^{\Phi}\}^{\mathrm{T}} = e^{\mathrm{T}\Phi}.$$

174. If Ξ and Φ are homologous dyadics, and such that

$$\Xi^2 . \Phi = -\Phi,$$

the definitions of No. 171 give immediately

$$e^{\Xi . \Phi} = \cos\Phi + \Xi \sin\Phi,$$
$$e^{-\Xi . \Phi} = \cos\Phi - \Xi \sin\Phi,$$

whence

$$\cos\Phi = \tfrac{1}{2}\{e^{\Xi\Phi} + e^{-\Xi\Phi}\},$$
$$\sin\Phi = -\tfrac{1}{2}\Xi\{e^{\Xi\Phi} - e^{-\Xi\Phi}\}.$$

175. If $\Phi.\Psi = \Psi.\Phi = 0$,

$$\{\Phi+\Psi\}^2 = \Phi^2 + \Psi^2, \qquad \{\Phi+\Psi\}^3 = \Phi^3 + \Psi^3, \text{ etc.}$$

Therefore

$$e^{\Phi+\Psi} = e^{\Phi} + e^{\Psi} - \mathrm{I},$$
$$\cos\{\Phi+\Psi\} = \cos\Phi + \cos\Psi - \mathrm{I},$$
$$\sin\{\Phi+\Psi\} = \sin\Phi + \sin\Psi.$$

176.

$$|e^{\Phi}| = e^{\Phi_s}.$$

For the first member of this equation is the limit of

$$|\{\mathrm{I}+\mathrm{N}^{-1}\Phi\}^{\mathrm{N}}|, \text{ that is, of } |\mathrm{I}+\mathrm{N}^{-1}\Phi|^{\mathrm{N}}.$$

If we set $\Phi = \alpha i + \beta j + \gamma k$, the limit becomes that of

$$(1+\mathrm{N}^{-1}\alpha.i+\mathrm{N}^{-1}\beta.j+\mathrm{N}^{-1}\gamma.k)^{\mathrm{N}}, \text{ or } (1+\mathrm{N}^{-1}\Phi_s)^{\mathrm{N}},$$

the limit of which is the second member of the equation to be proved.

177. By the definition of exponentials, the expression

$$e^{q\{kj-jk\}}$$

represents the limit of

$$\{\mathrm{I}+q\mathrm{N}^{-1}\{kj-jk\}\}^{\mathrm{N}}.$$

Now $\mathrm{I}+q\mathrm{N}^{-1}\{kj-jk\}$ evidently represents a versor having the axis i and the infinitesimal angle of version $q\mathrm{N}^{-1}$. Hence the above exponential represents a versor having the same axis and the angle of version q. If we set $qi=\omega$, the exponential may be written

$$e^{\mathrm{I}\times\omega}.$$

Such an expression therefore represents a versor. The axis and direction of rotation are determined by the direction of ω, and the angle of rotation is equal to the magnitude of ω. The value of the versor will not be affected by increasing or diminishing the magnitude of ω by 2π.

178. If, as in No. 151,

$$\Phi=a\alpha\alpha'+b\beta\beta'+c\gamma\gamma',$$

the definitions of No. 171 give

$$e^{\Phi}=e^{a}\alpha\alpha'+e^{b}\beta\beta'+e^{c}\gamma\gamma',$$

$$\cos\Phi=\cos a\,\alpha\alpha'+\cos b\,\beta\beta'+\cos c\,\gamma\gamma',$$

$$\sin\Phi=\sin a\,\alpha\alpha'+\sin b\,\beta\beta'+\sin c\,\gamma\gamma'.$$

If a, b, c are positive and unequal, we may add, by No. 172,

$$\log\Phi=\log a\,\alpha\alpha'+\log b\,\beta\beta'+\log c\,\gamma\gamma'.$$

179. If, as in No. 153,

$$\Phi=a\alpha\alpha'+b\{\beta\beta'+\gamma\gamma'\}+c\{\gamma\beta'-\beta\gamma'\}$$
$$=a\alpha\alpha'+p\cos q\{\beta\beta'+\gamma\gamma'\}+p\sin q\{\gamma\beta'-\beta\gamma'\},$$

we have by No. 173 $e^{\Phi}=e^{a\alpha\alpha'}\cdot e^{b\{\beta\beta'+\gamma\gamma'\}}\cdot e^{c\{\gamma\beta'-\beta\gamma'\}}$.

But

$$e^{a\alpha\alpha'}=e^{a}\alpha\alpha'+\beta\beta'+\gamma\gamma',$$

$$e^{b\{\beta\beta'+\gamma\gamma'\}}=\alpha\alpha'+e^{b}\{\beta\beta'+\gamma\gamma'\},$$

$$e^{c\{\gamma\beta'-\beta\gamma'\}}=\alpha\alpha'+\cos c\{\beta\beta'+\gamma\gamma'\}+\sin c\{\gamma\beta'-\beta\gamma'\}.$$

Therefore, $e^{\Phi}=e^{a}\alpha\alpha'+e^{b}\cos c\{\beta\beta'+\gamma\gamma'\}+e^{b}\sin c\{\gamma\beta'-\beta\gamma'\}$.

Hence, if a is positive,

$$\log\Phi=\log a\,\alpha\alpha'+\log p\{\beta\beta'+\gamma\gamma'\}+q\{\gamma\beta'-\beta\gamma'\}.$$

Since the value of Φ is not affected by increasing or diminishing q by 2π, the function $\log\Phi$ is many-valued.

To find the value of $\cos\Phi$ and $\sin\Phi$, let us set

$$\Theta=b\{\beta\beta'+\gamma\gamma'\}+c\{\gamma\beta'-\beta\gamma'\},$$
$$\Xi=\gamma\beta'-\beta\gamma'.$$

Then, by No, 175,

$$\cos\Phi=\cos\{a\alpha\alpha'\}+\cos\Theta-\mathrm{I}.$$

But $\cos\{a\alpha\alpha'\}-\mathrm{I}=\cos a\alpha\alpha'-a\alpha'.$

Therefore,

$$\cos\Phi=\cos a\alpha\alpha'-a\alpha'+\cos\Theta.$$

Now, by No. 174,

$$\cos\Theta = \tfrac{1}{2}\{e^{\Xi.\Theta} + e^{-\Xi.\Theta}\}.$$

Since

$$\Xi.\Theta = -c\{\beta\beta' + \gamma\gamma'\} + b\{\gamma\beta' - \beta\gamma'\},$$

$$e^{\Xi.\Theta} = \alpha\alpha' + e^{-c}\cos b\,\{\beta\beta' + \gamma\gamma'\} + e^{-c}\sin b\,\{\gamma\beta' - \beta\gamma'\},$$

$$e^{-\Xi.\Theta} = \alpha\alpha' + e^{c}\cos b\,\{\beta\beta' + \gamma\gamma'\} - e^{c}\sin b\,\{\gamma\beta' - \beta\gamma'\}.$$

Therefore

$$\cos\Theta = \alpha\alpha' + \tfrac{1}{2}(e^{c} + e^{-c})\cos b\,\{\beta\beta' + \gamma\gamma'\} - \tfrac{1}{2}(e^{c} - e^{-c})\sin b\,\{\gamma\beta' - \beta\gamma'\},$$

and

$$\cos\Phi = \cos a\,\alpha\alpha' + \tfrac{1}{2}(e^{c} + e^{-c})\cos b\,\{\beta\beta' + \gamma\gamma'\} - \tfrac{1}{2}(e^{c} - e^{-c})\sin b\,\{\gamma\beta' - \beta\gamma'\}.$$

In like manner we find

$$\sin\Phi = \sin a\,\alpha\alpha' + \tfrac{1}{2}(e^{c} + e^{-c})\sin b\,\{\beta\beta' + \gamma\gamma'\} + \tfrac{1}{2}(e^{c} - e^{-c})\cos b\,\{\gamma\beta' - \beta\gamma'\}.$$

180. If α, β, γ and α', β', γ' are reciprocals, and

$$\Phi = a\,\alpha\alpha' + b\{\beta\beta' + \gamma\gamma'\} + c\,\beta\gamma',$$

and N is any whole number,

$$\Phi^{N} = a^{N}\alpha\alpha' + b^{N}\{\beta\beta' + \gamma\gamma'\} + Nb^{N-1}c\,\beta\gamma'.$$

Therefore,

$$e^{\Phi} = e^{a}\alpha\alpha' + e^{b}\{\beta\beta' + \gamma\gamma'\} + e^{b}\,c\,\beta\gamma',$$

$$\cos\Phi = \cos a\,\alpha\alpha' + \cos b\,\{\beta\beta' + \gamma\gamma'\} - c\sin b\,\beta\gamma',$$

$$\sin\Phi = \sin a\,\alpha\alpha' + \sin b\,\{\beta\beta' + \gamma\gamma'\} + c\cos b\,\beta\gamma'.$$

If a and b are unequal, and c other than zero, we may add

$$\log\Phi = \log a\,\alpha\alpha' + \log b\{\beta\beta' + \gamma\gamma'\} + cb^{-1}\beta\gamma.$$

181. If α, β, γ, and α', β', γ' are reciprocals, and

$$\Phi = a\mathrm{I} + b\{\alpha\beta' + \beta\gamma'\} + c\,\alpha\gamma',$$

and N is a whole number,

$$\Phi^{N} = a^{N}\mathrm{I} + Na^{N-1}b\{\alpha\beta' + \beta\gamma'\} + (Na^{N-1}c + \tfrac{1}{2}N(N-1)a^{N-2}b^{2})\alpha\gamma'.$$

Therefore

$$e^{\Phi} = e^{a}\mathrm{I} + e^{a}\,b\{\alpha\beta' + \beta\gamma'\} + e^{a}(\tfrac{1}{2}b^{2} + c)\alpha\gamma',$$

$$\cos\Phi = \cos a\,\mathrm{I} - b\sin a\,\{\alpha\beta' + \beta\alpha'\} - (\tfrac{1}{2}b^{2}\cos a + c\sin a)\alpha\gamma',$$

$$\sin\Phi = \sin a\,\mathrm{I} + b\cos a\,\{\alpha\beta' + \beta\alpha'\} - (\tfrac{1}{2}b^{2}\sin a - c\cos a)\alpha\gamma'.$$

Unless $b = 0$, we may add

$$\log\Phi = \log a\,\mathrm{I} + ba^{-1}\{\alpha\beta' + \beta\alpha'\} + (ca^{-1} - \tfrac{1}{2}b^{2}a^{-2})\alpha\gamma'.$$

182. If we suppose any dyadic Φ to vary, but with the limitation that all its values are homologous, we may obtain from the definitions of No. 171

$$d\{e^{\Phi}\} = e^{\Phi}.d\Phi, \tag{1}$$

$$d\sin\Phi = \cos\Phi.d\Phi, \tag{2}$$

$$d\cos\Phi = -\sin\Phi.d\Phi, \tag{3}$$

$$d\log\Phi = \Phi^{-1}.d\Phi, \tag{4}$$

as in the ordinary calculus, but we must not apply these equations to cases in which the values of Φ are not homologous.

183. If, however, Γ is any constant dyadic, the variations of $t\Gamma$ will necessarily be homologous with $t\Gamma$, and we may write without other limitation than that Γ is constant,

$$\frac{d\{e^{t\Gamma}\}}{dt} = \Gamma . e^{t\Gamma} \tag{1}$$

$$\frac{d\sin\{t\Gamma\}}{dt} = \Gamma . \cos\{t\Gamma\}, \tag{2}$$

$$\frac{d\cos\{t\Gamma\}}{dt} = -\Gamma . \sin\{t\Gamma\}, \tag{3}$$

$$\frac{d\log\{t\Gamma\}}{dt} = \frac{\mathrm{I}}{t}. \tag{4}$$

A second differentiation gives

$$\frac{d^2\{e^{t\Gamma}\}}{dt^2} = \Gamma^2 . e^{t\Gamma}, \tag{5}$$

$$\frac{d^2\sin\{t\Gamma\}}{dt^2} = -\Gamma^2 . \sin\{t\Gamma\}, \tag{6}$$

$$\frac{d^2\cos\{t\Gamma\}}{dt^2} = -\Gamma^2 . \cos\{t\Gamma\}. \tag{7}$$

184. It follows that if we have a differential equation of the form

$$\frac{d\rho}{dt} = \Gamma . \rho,$$

the integral equation will be of the form

$$\rho = e^{t\Gamma} . \rho',$$

ρ' representing the value of ρ for $t = 0$. For this gives

$$\frac{d\rho}{dt} = \Gamma . e^{t\Gamma} . \rho' = \Gamma . \rho,$$

and the proper value of ρ for $t = 0$.

185. *Def.*—A flux which is a linear function of the position-vector is called a *homogeneous-strain-flux* from the nature of the strain which it produces. Such a flux may evidently be represented by a dyadic.

In the equations of the last paragraph, we may suppose ρ to represent a position-vector, t the time, and Γ a homogeneous-strain-flux. Then $e^{t\Gamma}$ will represent the strain produced by the flux Γ in the time t.

In like manner, if Λ represents a homogeneous strain, $\{\log \Lambda\}/t$ will represent a homogeneous-strain-flux which would produce the strain Λ in the time t.

186. If we have

$$\frac{d^2\rho}{dt^2}=\Gamma^2.\rho,$$

where Γ is complete, the integral equation will be of the form

$$\rho=e^{t\Gamma}.\alpha+e^{-t\Gamma}.\beta.$$

For this gives

$$\frac{d\rho}{dt}=\Gamma.e^{t\Gamma}.\alpha-\Gamma.e^{-t\Gamma}.\beta,$$

$$\frac{d^2\rho}{dt^2}=\Gamma^2.e^{t\Gamma}.\alpha+\Gamma^2.e^{-t\Gamma}.\beta=\Gamma^2.\rho,$$

and α and β may be determined so as to satisfy the equations

$$\rho_{t=0}=\alpha+\beta,$$

$$\left[\frac{d\rho}{dt}\right]_{t=0}=\Gamma.\{\alpha-\beta\}.$$

187. The differential equation

$$\frac{d^2\rho}{dt^2}=-\Gamma^2.\rho$$

will be satisfied by

$$\rho=\cos\{t\Gamma\}.\alpha+\sin\{t\Gamma\}.\beta,$$

whence

$$\frac{d\rho}{dt}=-\Gamma.\sin\{t\Gamma\}.\alpha+\Gamma.\cos\{t\Gamma\}.\beta,$$

$$\frac{d^2\rho}{dt^2}=-\Gamma^2.\cos\{t\Gamma\}.\alpha-\Gamma^2.\sin\{t\Gamma\}.\beta=-\Gamma^2.\rho.$$

If Γ is complete, the constants α and β may be determined to satisfy the equations

$$\rho_{t=0}=\alpha,$$

$$\left[\frac{d\rho}{dt}\right]_{t=0}=\Gamma.\beta.$$

188. If

$$\frac{d^2\rho}{dt^2}=\{\Gamma^2-\Lambda^2\}.\rho,$$

where $\Gamma^2-\Lambda^2$ is a complete dyadic, and

$$\Gamma.\Lambda=\Lambda.\Gamma=0,$$

we may set

$$\rho=\left\{\tfrac{1}{2}e^{t\Gamma}+\tfrac{1}{2}e^{-t\Gamma}+\cos\{t\Lambda\}-\mathrm{I}\right\}.\alpha+\left\{\tfrac{1}{2}e^{t\Gamma}-\tfrac{1}{2}e^{-t\Gamma}+\sin\{t\Lambda\}\right\}.\beta,$$

which gives

$$\frac{d\rho}{dt}=\left\{\tfrac{1}{2}\Gamma.e^{t\Gamma}-\tfrac{1}{2}\Gamma.e^{-t\Gamma}-\Lambda.\sin\{t\Lambda\}\right\}.\alpha+\left\{\tfrac{1}{2}\Gamma.e^{t\Gamma}+\tfrac{1}{2}\Gamma.e^{-t\Gamma}+\Lambda.\cos\{t\Lambda\}\right\}.\beta,$$

$$\frac{d^2\rho}{dt^2}=\left\{\tfrac{1}{2}\Gamma^2.e^{t\Gamma}+\tfrac{1}{2}\Gamma^2.e^{-t\Gamma}-\Lambda^2.\cos\{t\Lambda\}\right\}.\alpha+\left\{\tfrac{1}{2}\Gamma^2.e^{t\Gamma}-\tfrac{1}{2}\Gamma^2.e^{-t\Gamma}-\Lambda^2.\sin\{t\Lambda\}\right\}.\beta$$

$$=\{\Gamma^2-\Lambda^2\}.\rho.$$

The constants α and β are to be determined by

$$\rho_{t=0} = \alpha,$$

$$\left[\frac{d\rho}{dt}\right]_{t=0} = \{\Gamma + \Lambda\}.\beta.$$

189. It will appear, on reference to Nos. 155–157, that every complete dyadic may be expressed in one of three forms, viz., as a square, as a square with the negative sign, or as a difference of squares of two dyadics of which both the direct products are equal to zero. It follows that every equation of the form

$$\frac{d^2\rho}{dt^2} = \Theta.\rho,$$

where Θ is any constant and complete dyadic, may be integrated by the preceding formulæ.

NOTE ON BIVECTOR ANALYSIS.*

1. A vector is determined by three algebraic quantities. It often occurs that the solution of the equations by which these are to be determined gives imaginary values, i.e., instead of scalars we obtain biscalars, or expressions of the form $a + \iota b$, where a and b are scalars, and $\iota = \sqrt{-1}$. It is most simple, and always allowable, to consider the vector as determined by its components parallel to a normal system of axes. In other words, a vector may be represented in the form

$$xi + yj + zk.$$

Now if the vector is required to satisfy certain conditions, the solution of the equations which determine the values of x, y, and z, in the most general case, will give results of the form

$$x = x_1 + \iota x_2,$$

$$y = y_1 + \iota y_2,$$

$$z = z_1 + \iota z_2,$$

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* Thus far, in accordance with the purpose expressed in the footnote on page 17, we have considered only real values of scalars and vectors. The object of this limitation has been to present the subject in the most elementary manner. The limitation is however often inconvenient, and does not allow the most symmetrical and complete development of the subject in many important directions. Thus in Chapter V, and the latter part of Chapter III, the exclusion of imaginary values has involved a considerable sacrifice of simplicity both in the enunciation of theorems and in their demonstration. The student will find an interesting and profitable exercise in working over this part of the subject with the aid of imaginary values, especially in the discussion of the imaginary roots of the cubic equation on page 71, and in the use of the formula

$$e^{\iota\Phi} = \cos\Phi + \iota\sin\Phi$$

in developing the properties of the *sines*, *cosines*, and *exponentials* of dyadics.

where x_1, x_2, y_1, y_2, z_1, z_2 are scalars. Substituting these values in

$$xi+yj+zk,$$

we obtain $$(x_1+\iota x_2)i+(y_1+\iota y_2)j+(z_1+\iota z_2)k;$$

or, if we set

$$\rho_1=x_1i+y_1j+z_1k,$$

$$\rho_2=x_2i+y_2j+z_2k,$$

we obtain $$\rho_1+\iota\rho_2.$$

We shall call this a *bivector*, a term which will include a vector as a particular case. When we wish to express a bivector by a single letter, we shall use the small German letters. Thus we may write

$$\mathfrak{r}=\rho_1+\iota\rho_2.$$

An important case is that in which ρ_1 and ρ_2 have the same direction. The bivector may then be expressed in the form $(a+\iota b)\rho$, in which the vector factor, if we choose, may be a unit vector. In this case, we may say that the bivector has a *real direction*. In fact, if we express the bivector in the form

$$(x_1+\iota x_2)i+(y_1+\iota y_2)j+(z_1+\iota z_2)k$$

the ratios of the coefficients of i, j, and k, which determine the direction cosines of the vector, will in this case be real.

2. The consideration that operations upon bivectors may be regarded as operations upon their biscalar x-, y- and z-components is sufficient to show the possibility of a bivector analysis and to indicate what its rules must be. But this point of view does not afford the most simple conception of the operations which we have to perform upon bivectors. It is desirable that the definitions of the fundamental operations should be independent of such extraneous considerations as any system of axes.

The various signs of our analysis, when applied to bivectors, may therefore be defined as follows, viz.,

The bivector equation

$$\mu'+\iota\nu'=\mu''+\iota\nu''$$

implies the two vector equations

$$\mu'=\mu'', \quad \text{and} \quad \nu'=\nu''.$$

$$-[\mu+\iota\nu]=-\mu+\iota[-\nu].$$

$$[\mu'+\iota\nu']+[\mu''+\iota\nu'']=[\mu'+\mu'']+\iota[\nu'+\nu''].*$$

$$[\mu'+\iota\nu'].[\mu''+\iota\nu'']=[\mu'.\mu''-\nu'.\nu'']+\iota[\mu'.\nu''+\nu'.\mu''].$$

$$[\mu'+\iota\nu']\times[\mu''+\iota\nu'']=[\mu'\times\mu''-\nu'\times\nu'']+\iota[\mu'\times\nu''+\nu'\times\mu''].$$

---

* $(a+\iota b)[\mu+\iota\nu]=a\mu-b\nu+\iota[a\nu+b\mu]$.
$[\mu+\iota\nu](a+\iota b)=\mu a-\nu b+\iota[\mu b+\nu a]$.
Therefore the position of the scalar factor is indifferent. [MS. note by author.]

With these definitions, a great part of the laws of vector analysis may be applied at once to bivector expressions. But an equation which is impossible in vector analysis may be possible in bivector analysis, and in general the number of roots of an equation, or of the values of a function, will be different according as we recognize, or do not recognize, imaginary values.

3. *Def.*—Two bivectors, or two biscalars, are said to be *conjugate*, when their real parts are the same, and their imaginary parts differ in sign, and in sign only.

Hence, the product of the conjugates of any number of bivectors and biscalars is the conjugate of the product of the bivectors and biscalars. This is true of any kind of product.

The products of a vector and its conjugate are as follows:

$$[\mu+\iota\nu].[\mu-\iota\nu]=\mu.\mu+\nu.\nu$$

$$[\mu+\iota\nu]\times[\mu-\iota\nu]=2\iota\nu\times\mu$$

$$[\mu+\iota\nu][\mu-\iota\nu]=\{\mu\mu+\nu\nu\}+\iota\{\nu\mu-\mu\nu\}.$$

Hence, if μ and $\iota\nu$ represent the real and imaginary parts of a bivector, the values of

$$\mu.\mu+\nu.\nu, \qquad \mu\times\nu, \qquad \mu\mu+\nu\nu, \qquad \nu\mu-\mu\nu,$$

are not affected by multiplying the bivector by a biscalar of the form $a+\iota b$, in which $a^2+b^2=1$, say a *cyclic scalar*. Thus, if we set

$$\mu'+\iota\nu'=(a+\iota b)[\mu+\iota\nu],$$

we shall have

$$\mu'-\iota\nu'=(a-\iota b)[\mu-\iota\nu],$$

and

$$[\mu'+\iota\nu'].[\mu'-\iota\nu']=[\mu+\iota\nu].[\mu-\iota\nu].$$

That is,

$$\mu'.\mu'+\nu'.\nu'=\mu.\mu+\nu.\nu;$$

and so in the other cases.

4. *Def.*—In biscalar analysis, the product of a biscalar and its conjugate is a positive scalar. The positive square root of this scalar is called the modulus of the biscalar. In bivector analysis, the direct product of a bivector and its conjugate is, as seen above, a positive scalar. The positive square root of this scalar may be called the *modulus* of the *bivector*. When this modulus vanishes, the bivector vanishes, and only in this case. If the bivector is multiplied by a biscalar, its modulus is multiplied by the modulus of the biscalar. The conjugate of a (real) vector is the vector itself, and the modulus of the vector is the same as its magnitude.

5. *Def.*—If between two vectors, α and β, there subsists a relation of the form

$$\alpha=n\beta,$$

where n is a scalar, we say that the vectors are parallel. Analogy leads us to call two bivectors *parallel*, when there subsists between them a relation of the form

$$\mathfrak{a} = m\mathfrak{b},$$

where m (in the most general case) is a biscalar.

To aid us in comprehending the geometrical signification of this relation, we may regard the biscalar as consisting of two factors, one of which is a positive scalar (the modulus of the biscalar), and the other may be put in the form $\cos q + \iota \sin q$. The effect of multiplying a bivector by a positive scalar is obvious. To understand the effect of a multiplier of the form $\cos q + \iota \sin q$ upon a bivector $\mu + \iota\nu$, let us set

$$\mu' + \iota\nu' = (\cos q + \iota \sin q)[\mu + \iota\nu].$$

We have then

$$\mu' = \cos q\, \mu - \sin q\, \nu,$$

$$\nu' = \cos q\, \nu + \sin q\, \mu.$$

Now if μ and ν are of the same magnitude and at right angles, the effect of the multiplication is evidently to rotate these vectors in their plane an angular distance q, which is to be measured in the direction from ν to μ. In any case we may regard μ and ν as the projections (by parallel lines) of two perpendicular vectors of the same length. The two last equations show that μ' and ν' will be the projections of the vectors obtained by the rotation of these perpendicular vectors in their plane through the angle q. Hence, if we construct an ellipse of which μ and ν are conjugate semi-diameters, μ' and ν' will be another pair of conjugate semi-diameters, and the sectors between μ and μ' and between ν and ν', will each be to the whole area of the ellipse as q to 2π, the sector between ν and ν' lying on the same side of ν and μ, and that between μ and μ' lying on the same side of μ as $-\nu$.

It follows that any bivector $\mu + \iota\nu$ may be put in the form

$$(\cos q + \iota \sin q)[\alpha + \iota\beta],$$

in which α and β are at right angles, being the semi-axes of the ellipse of which μ and ν are conjugate semi-diameters. This ellipse we may call the *directional ellipse* of the bivector. In the case of a real vector, or of a vector having a real direction, it reduces to a straight line. In any other case, the angular direction from the imaginary to the real part of the bivector is to be regarded as positive in the ellipse, and the specification of the ellipse must be considered incomplete without the indication of this direction.

Parallelism of bivectors, then, signifies the similarity and similar position of their directional ellipses. Similar position includes identity of the angular directions mentioned above.

6. To reduce a given bivector $\mathfrak{r}$ to the above form, we may set

$$\begin{aligned}\mathfrak{r}.\mathfrak{r} &= (\cos q + \iota \sin q)^2 [\alpha + \iota\beta].[\alpha + \iota\beta] \\ &= (\cos 2q + \iota \sin 2q)\,(\alpha.\alpha - \beta.\beta) \\ &= a + \iota b,\end{aligned}$$

where a and b are scalars, which we may regard as known. The value of q may be determined by the equation

$$\tan 2q = \frac{b}{a},$$

the quadrant to which $2q$ belongs being determined so as to give $\sin 2q$ and $\cos 2q$ the same signs as b and a. Then α and β will be given by the equation

$$\alpha + \iota\beta = (\cos q - \iota \sin q)\mathfrak{r}.$$

The solution is indeterminate when the real and imaginary parts of the given bivector are perpendicular and equal in magnitude. In this case the directional ellipse is a circle, and the bivector may be called *circular*. The criterion of a circular bivector is

$$\mathfrak{r}.\mathfrak{r} = 0.$$

It is especially to be noticed that from this equation we cannot conclude that

$$\mathfrak{r} = 0,$$

as in the analysis of real vectors. This may also be shown by expressing $\mathfrak{r}$ in the form $xi + yj + zk$, in which x, y, z are biscalars. The equation then becomes

$$x^2 + y^2 + z^2 = 0,$$

which evidently does not require x, y, and z to vanish, as would be the case if only real values are considered.

7. *Def.*—We call two vectors ρ and σ perpendicular when $\rho.\sigma = 0$. Following the same analogy, we shall call two bivectors $\mathfrak{r}$ and $\mathfrak{s}$ perpendicular, when

$$\mathfrak{r}.\mathfrak{s} = 0.$$

In considering the geometrical signification of this equation, we shall first suppose that the real and imaginary components of $\mathfrak{r}$ and $\mathfrak{s}$ lie in the same plane, and that both $\mathfrak{r}$ and $\mathfrak{s}$ have not real directions. It is then evidently possible to express them in the form

$$m[\alpha + \iota\beta], \qquad m'[\alpha' + \iota\beta'],$$

where m and m' are biscalar, α and β are at right angles, and α' parallel with β. Then the equation $\mathfrak{r}.\mathfrak{s} = 0$ requires that

$$\beta.\beta' = 0, \quad \text{and} \quad \alpha.\beta' + \beta.\alpha' = 0.$$

This shows that the directional ellipses of the two bivectors are similar and the angular direction from the real to the imaginary

component is the same in both, but the major axes of the ellipses are perpendicular. The case in which the directions of $\mathfrak{r}$ and $\mathfrak{s}$ are real, forms no exception to this rule.

It will be observed that every circular bivector is perpendicular to itself, and to every parallel bivector.

If two bivectors, $\mu+\iota\nu$, $\mu'+\iota\nu'$, which do not lie in the same plane are perpendicular, we may resolve μ and ν into components parallel and perpendicular to the plane of μ' and ν'. The components perpendicular to the plane evidently contribute nothing to the value of

$$[\mu+\iota\nu].[\mu'+\iota\nu'].$$

Therefore the components of μ and ν parallel to the plane of μ', ν', form a bivector which is perpendicular to $\mu'+\iota\nu'$. That is, if two bivectors are perpendicular, the directional ellipse of either, projected upon the plane of the other and rotated through a quadrant in that plane, will be similar and similarly situated to the directional ellipse of the second.

8. A bivector may be divided in one and only one way into parts parallel and perpendicular to another, provided that the second is not circular. If $\mathfrak{a}$ and $\mathfrak{b}$ are the bivectors, the parts of $\mathfrak{a}$ will be

$$\frac{\mathfrak{b}.\mathfrak{a}}{\mathfrak{b}.\mathfrak{b}}\mathfrak{b} \quad \text{and} \quad \mathfrak{a}-\frac{\mathfrak{b}.\mathfrak{a}}{\mathfrak{b}.\mathfrak{b}}\mathfrak{b}.$$

If $\mathfrak{b}$ is circular, the resolution of $\mathfrak{a}$ is impossible, unless it is perpendicular to $\mathfrak{b}$. In this case the resolution is indeterminate.

9. Since $\mathfrak{a}\times\mathfrak{b}.\mathfrak{a}=0$, and $\mathfrak{a}\times\mathfrak{b}.\mathfrak{b}=0$, $\mathfrak{a}\times\mathfrak{b}$ is perpendicular to $\mathfrak{a}$ and $\mathfrak{b}$. We may regard the plane of the product as determined by the condition that the directional ellipses of the factors projected upon it become similar and similarly situated. The directional ellipse of the product is similar to these projections, but its orientation is different by 90°. It may easily be shown that $\mathfrak{a}\times\mathfrak{b}$ vanishes only with $\mathfrak{a}$ or $\mathfrak{b}$, or when $\mathfrak{a}$ and $\mathfrak{b}$ are parallel.

10. The bivector equation

$$(\mathfrak{a}\times\mathfrak{b}.\mathfrak{c})\mathfrak{b}-(\mathfrak{b}.\mathfrak{c}\times\mathfrak{b})\mathfrak{a}+(\mathfrak{c}.\mathfrak{b}\times\mathfrak{a})\mathfrak{b}-(\mathfrak{b}.\mathfrak{a}\times\mathfrak{b})\mathfrak{c}=0$$

is identical, as may be verified by substituting expressions of the form $xi+yj+zk$ (x, y, z being biscalars), for each of the bivectors. (Compare No. 37.) This equation shows that if the product $\mathfrak{a}\times\mathfrak{b}$ of any two bivectors vanishes, one of these will be equal to the other with a biscalar coefficient, that is, they will be parallel, according to the definition given above. If the product $\mathfrak{a}.\mathfrak{b}\times\mathfrak{c}$ of any three bivectors vanishes, the equation shows that one of these may be expressed as a sum of the other two with biscalar coefficients. In this case, we may say (from the analogy of the scalar analysis) that the three bivectors are complanar. (This does not imply that they

lie in any same *real* plane.) If $\mathfrak{a}.\mathfrak{b}\times\mathfrak{c}$ is not equal to zero, the equation shows that any fourth bivector may be expressed as a sum of $\mathfrak{a}$, $\mathfrak{b}$, and $\mathfrak{c}$ with biscalar coefficients, and indicates how these coefficients may be determined.

11. The equation

$$(\mathfrak{r}.\mathfrak{a})\mathfrak{b}\times\mathfrak{c}+(\mathfrak{r}.\mathfrak{b})\mathfrak{c}\times\mathfrak{a}+(\mathfrak{r}.\mathfrak{c})\mathfrak{a}\times\mathfrak{b}=(\mathfrak{a}\times\mathfrak{b}.\mathfrak{c})\mathfrak{r}$$

is also identical, as may easily be verified. If we set

$$\mathfrak{c}=\mathfrak{a}\times\mathfrak{b},$$

and suppose that

$$\mathfrak{r}.\mathfrak{a}=0, \quad \mathfrak{r}.\mathfrak{b}=0,$$

the equation becomes

$$(\mathfrak{r}.\mathfrak{a}\times\mathfrak{b})\mathfrak{a}\times\mathfrak{b}=(\mathfrak{a}\times\mathfrak{b}.\mathfrak{a}\times\mathfrak{b})\mathfrak{r}.$$

This shows that if a bivector $\mathfrak{r}$ is perpendicular to two bivectors $\mathfrak{a}$ and $\mathfrak{b}$, which are not parallel, $\mathfrak{r}$ will be parallel to $\mathfrak{a}\times\mathfrak{b}$. Therefore all bivectors which are perpendicular to two given bivectors are parallel to each other, unless the given two are parallel.

[*Note by Editors.*—The notation $|\Phi|\,\Phi_c^{-1}$, used on page 64, was later improved by the author by the introduction of his Double Multiplication, according to which the above expression is represented by Φ_2, and $|\Phi|$ by Φ_3. See this volume, pages 112, 160, and 181. For an extended treatment of Professor Gibbs's researches on Double Multiplication in their application to Vector Analysis see pp. 306-321, and 333 of "Vector Analysis," by E. B. Wilson, Chas. Scribner's Sons, New York, 1901.]

IV.

ON MULTIPLE ALGEBRA.

ADDRESS BEFORE THE SECTION OF MATHEMATICS AND ASTRONOMY OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, BY THE VICE-PRESIDENT.

[*Proceedings of the American Association for the Advancement of Science*, vol. XXXV. pp. 37-66, 1886.]

IT has been said that "the human mind has never invented a labor-saving machine equal to algebra."* If this be true, it is but natural and proper that an age like our own, characterized by the multiplication of labor-saving machinery, should be distinguished by an unexampled development of this most refined and most beautiful of machines. That such has been the case, none will question. The improvement has been in every part. Even to enumerate the principal lines of advance would be a task for any one; for me an impossibility. But if we should ask, in what direction the advance has been made which is to characterize the development of algebra in our day, we may, I think, point to that broadening of its field and methods which gives us *multiple algebra*.

Of the importance of this change in the conception of the office of algebra, it is hardly necessary to speak: that it is really characteristic of our time will be most evident if we go back some two or three-score years, to the time when the seeds were sown which are now yielding so abundant a harvest. The failure of Möbius, Hamilton, Grassmann, Saint-Venant to make an immediate impression upon the course of mathematical thought in any way commensurate with the importance of their discoveries is the most conspicuous evidence that the times were not ripe for the methods which they sought to introduce. A satisfactory theory of the imaginary quantities of ordinary algebra, which is essentially a simple case of multiple algebra, with difficulty obtained recognition in the first third of this century. We must observe that this *double algebra*, as it has been called, was not sought for or invented;—it forced itself, unbidden, upon the attention of mathematicians, and with its rules already formed.

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* *The Nation*, vol. xxxiii, p. 237.

But the idea of double algebra, once received, although as it were unwillingly, must have suggested to many minds more or less distinctly the possibility of other multiple algebras, of higher orders, possessing interesting or useful properties.

The application of double algebra to the geometry of the plane suggested not unnaturally to Hamilton the idea of a triple algebra which should be capable of a similar application to the geometry of three dimensions. He was unable to find a satisfactory triple algebra, but discovered at length a quadruple algebra, *quaternions*, which answered his purpose, thus satisfying, as he says in one of his letters, an intellectual want which had haunted him at least fifteen years. So confident was he of the value of this algebra, that the same hour he obtained permission to lay his discovery before the Royal Irish Academy, which he did on November 13, 1843.* This system of multiple algebra is far better known than any other, except the ordinary double algebra of imaginary quantities,—far too well known to require any especial notice at my hands. All that here requires our attention is the close historical connection between the imaginaries of ordinary algebra and Hamilton's system, a fact emphasized by Hamilton himself and most writers on quaternions. It was quite otherwise with Möbius and Grassmann.

The point of departure of the *Barycentrischer Calcul* of Möbius, published in 1827,—a work of which Clebsch has said that it can never be admired enough,†—is the use of equations in which the terms consist of letters representing points with numerical coefficients, to express barycentric relations between the points. Thus, that the point S is the centre of gravity of weights, a, b, c, d, placed at the points A, B, C, D, respectively, is expressed by the equation

$$(a+b+c+d)S=aA+bB+cC+dD.$$

An equation of the more general form

$$aA+bB+cC+\text{etc.}=pP+qQ+rR+\text{etc.}$$

signifies that the weights a, b, c, etc., at the points A, B, C, etc., have the same sum and the same centre of gravity as the weights p, q, r, etc., at the points P, Q, R, etc., or, in other words, that the former are barycentrically equivalent to the latter. Such equations, of which each represents four ordinary equations, may evidently be multiplied or divided by scalars,‡ may be added or subtracted, and may have

* *Phil. Mag.* (3), vol. xxv, p. 490; *North British Review*, vol. xlv (1866), p. 57.

† See his eulogy on Plücker, p. 14, *Gött. Abhandl.*, vol. xvi.

‡ I use this term in Hamilton's sense, to denote the ordinary positive and negative quantities of algebra. It may, however, be observed that in most cases in which I shall have occasion to use it, the proposition would hold without exclusion of imaginary quantities,—that this exclusion is generally for simplicity and not from necessity.

their terms arranged and transposed, exactly like the ordinary equations of algebra. It follows that the elimination of letters representing points from equations of this kind is performed by the rules of ordinary algebra. This is evidently the beginning of a quadruple algebra, and is identical, as far as it goes, with Grassmann's marvellous geometrical algebra.

In the same work we find also, for the first time so far as I am aware, the distinction of positive and negative consistently carried out on the designation of segments of lines, of triangles, and of tetrahedra, viz., that a change in place of two letters, in such expressions as AB, ABC, $ABCD$, is equivalent to prefixing the negative sign. It is impossible to overestimate the importance of this step, which gives to designations of this kind the generality and precision of algebra.

Moreover, if A, B, C are three points in the same straight line, and D any point outside of that line, the author observes that we have

$$AB+BC+CA=0,$$

and also, with D prefixed,

$$DAB+DBC+DCA=0.$$

Again, if A, B, C, D are four points in the same plane, and E any point outside of that plane, we have

$$ABC-BCD+CDA-DAB=0,$$

and also, with E prefixed,

$$EABC-EBCD+ECDA-EDAB=0.$$

The similarity to multiplication in the derivation of these formulæ cannot have escaped the author's notice. Yet he does not seem to have been able to generalize these processes. It was reserved for the genius of Grassmann to see that AB might be regarded as the product of A and B, DAB as the product of D and AB, and $EABC$ as the product of E and ABC. That Möbius could not make this step was evidently due to the fact that he had not the conception of the addition of other multiple quantities than such as may be represented by masses situated at points. Even the addition of vectors (i.e., the fact that the composition of directed lines could be treated as an addition) seems to have been unknown to him at this time, although he subsequently discovered it, and used it in his *Mechanik des Himmels*, which was published in 1843. This addition of vectors, or *geometrical addition*, seems to have occurred independently to many persons.

Seventeen years after the *Barycentrischer Calcul*, in 1844, the year in which Hamilton's first papers on quaternions appeared in print, Grassmann published his *Lineale Ausdehnungslehre*, in which he developed the idea and the properties of the *external* or *combinatorial*

product, a conception which is perhaps to be regarded as the greatest monument of the author's genius. This volume was to have been followed by another, of the nature of which some intimation was given in the preface and in the work itself. We are especially told that the *internal product*,* which for vectors is identical except in sign with the scalar part of Hamilton's product (just as Grassmann's external product of two vectors is practically identical with the vector part of Hamilton's product), and the *open product*,† which in the language of to-day would be called a matrix, were to be treated in the second volume. But both the internal product of vectors and the open product are clearly defined, and their fundamental properties indicated, in this first volume.

This remarkable work remained unnoticed for more than twenty years, a fact which was doubtless due in part to the very abstract and philosophical manner in which the subject was presented. In consequence of this neglect the author changed his plan, and instead of a supplementary volume published in 1862 a single volume entitled *Ausdehnungslehre*, in which were treated, in an entirely different style, the same topics as in the first volume, as well as those which he had reserved for the second.

Deferring for the moment the discussion of these topics in order to follow the course of events, we find in the year following the first *Ausdehnungslehre* a remarkable memoir of Saint-Venant ‡, in which are clearly described the addition both of vectors and of oriented areas, the differentiation of these with respect to a scalar quantity, and a multiplication of two vectors and of a vector and an oriented area. These multiplications, called by the author *geometrical*, are entirely identical with Grassmann's external multiplication of the same quantities.

It is a striking fact in the history of the subject, that the short period of less than two years was marked by the appearance of well-developed and valuable systems of multiple algebra by British, German, and French authors, working apparently entirely independently of one another. No system of multiple algebra had appeared before, so far as I know, except such as were confined to additive processes with multiplication by scalars, or related to the ordinary double algebra of imaginary quantities. But the appearance of a single one of these systems would have been sufficient to mark an epoch, perhaps the most important epoch in the history of the subject.

In 1853 and 1854, Cauchy published several memoirs on what he called *clefs algébriques*.§ These were units subject generally to

* See the preface.
† See § 172.
‡ *Comptes Rendus*, vol. xxi, p. 620.
§ *Comptes Rendus*, vols. xxxvi, ff.

combinatorial multiplication. His principal application was to the theory of elimination. In this application, as in the law of multiplication, he had been anticipated by Grassmann.

We come next to Cayley's celebrated *Memoir on the Theory of Matrices* * in 1858, of which Sylvester has said that it seems to him to have ushered in the reign of Algebra the Second. † I quote this dictum of a master as showing his opinion of the importance of the subject and of the memoir. But the foundations of the theory of matrices, regarded as multiple quantities, seem to me to have been already laid in the *Ausdehnungslehre* of 1844. To Grassmann's treatment of this subject we shall recur later.

After the *Ausdehnungslehre* of 1862, already mentioned, we come to Hankel's *Vorlesungen über die complexen Zahlen*, 1867. Under this title the author treats of the imaginary quantities of ordinary algebra, of what he calls *alternirende Zahlen*, and of quaternions. These alternate numbers, like Cauchy's *clefs*, are quantities subject to Grassmann's law of combinatorial multiplication. This treatise, published twenty-three years after the first *Ausdehnungslehre*, marks the first impression which we can discover of Grassmann's ideas upon the course of mathematical thought. The transcendent importance of these ideas was fully appreciated by the author, whose very able work seems to have had considerable influence in calling the attention of mathematicians to the subject.

In 1870, Professor Benjamin Peirce published his *Linear Associative Algebra*, subsequently developed and enriched by his son, Professor C. S. Peirce. The fact that the edition was lithographed seems to indicate that even at this late date a work of this kind could only be regarded as addressed to a limited number of readers. But the increasing interest in such subjects is shown by the republication of this memoir in 1881,‡ as by that of the first *Ausdehnungslehre* in 1878.

The article on quaternions which has just appeared in the *Encyclopædia Britannica* mentions twelve treatises, including second editions and translations, besides the original treatises of Hamilton. That all the twelve are later than 1861 and all but two later than 1872 shows the rapid increase of interest in this subject in the last years.

Finally, we arrive at the *Lectures on the Principles of Universal Algebra* by the distinguished foreigner whose sojourn among us has given such an impulse to mathematical study in this country. The publication of these lectures, commenced in 1884 in the *American Journal of Mathematics*, has not as yet been completed,—a want but imperfectly supplied by the author's somewhat desultory publication

* *Phil. Trans.*, vol. cxlviii. † *Amer. Journ. Math.*, vol. vi, p. 271.

‡ *Amer. Journ. Math.*, vol. iv.

of many remarkable papers on the same subject (which might be more definitely expressed as the algebra of matrices) in various foreign journals.

It is not an accident that this century has seen the rise of multiple algebra. The course of the development of ideas in algebra and in geometry, although in the main independent of any aid from this source, has nevertheless to a very large extent been of a character which can only find its natural expression in multiple algebra.

Our Modern Higher Algebra is especially occupied with the theory of linear transformations. Now what are the first notions which we meet in this theory? We have a set of n variables, say x, y, z, and another set, say x', y', z', which are homogeneous linear functions of the first, and therefore expressible in terms of them by means of a block of n^2 coefficients. Here the quantities occur by sets, and invite the notations of multiple algebra. It was in fact shown by Grassmann in his first *Ausdehnungslehre* and by Cauchy nine years later, that the notations of multiple algebra afford a natural key to the subject of elimination.

Now I do not merely mean that we may save a little time or space by writing perhaps ρ for x, y and z; ρ' for x', y' and z'; and Φ for a block of n^2 quantities. But I mean that the subject as usually treated under the title of determinants has a stunted and misdirected development on account of the limitations of single algebra. This will appear from a very simple illustration. After a little preliminary matter the student comes generally to a chapter entitled "Multiplication of Determinants," in which he is taught that the product of the determinants of two matrices may be found by performing a somewhat lengthy operation on the two matrices, by which he obtains a third matrix, and then taking the determinant of this. But what significance, what value has this theorem? For aught that appears in the majority of treatises which I have seen, we have only a complicated and lengthy way of performing a simple operation. The real facts of the case may be stated as follows:

Suppose the set of n quantities ρ' to be derived from the set ρ by the matrix Φ, which we may express by

$$\rho' = \Phi . \rho ;$$

and suppose the set ρ'' to be derived from the set ρ' by the matrix Ψ, i.e.,

$$\rho'' = \Psi . \rho'.$$

and

$$\rho'' = \Psi . \Phi . \rho ;$$

it is evident that ρ'' can be derived from ρ by the operation of a single matrix, say Θ, i.e.,

$$\rho'' = \Theta . \rho,$$

so that

$$\Theta = \Psi . \Phi .$$

In the language of multiple algebra Θ is called the product of Ψ and Φ. It is of course interesting to see how it is derived from the latter, and it is little more than a schoolboy's exercise to determine this. Now this matrix Θ has the property that its determinant is equal to the products of the determinants of Ψ and Φ. And this property is all that is generally stated in the books, and the fundamental property, which is all that gives the subject its interest, that Θ is itself the product of Ψ and Φ in the language of multiple algebra, i.e., that operating by Θ is equivalent to operating successively by Φ and Ψ, is generally omitted. The chapter on this subject, in most treatises which I have seen, reads very like the play of Hamlet with Hamlet's part left out.

And what is the cause of this omission? Certainly not ignorance of the property in question. The fact that it is occasionally given would be a sufficient bar to this answer. It is because the author fails to see that his real subject is matrices and not determinants. Of course, in a certain sense, the author has a right to choose his subject. But this does not mean that the choice is unimportant, or that it should be determined by chance or by caprice. The problem well put is half solved, as we all know. If one chooses the subject ill, it will develop itself in a cramped manner.

But the case is really much worse than I have stated it. Not only is the true significance of the formation of Θ from Ψ and Φ not given, but the student is often not taught to form the matrix which is the product of Ψ and Φ, but one which is the product of one of these matrices and the conjugate of the other. Thus the proposition which is proved loses all its simplicity and significance, and must be recast before the instructor can explain its true bearings to the student. This fault has been denounced by Sylvester, and if anyone thinks I make too much of the standpoint from which the subject is viewed, I will refer him to the opening paragraphs of the "Lectures on Universal Algebra" in the sixth volume of the *American Journal of Mathematics*, where, with a wealth of illustration and an energy of diction which I cannot emulate, the most eloquent of mathematicians expresses his sense of the importance of the substitution of the idea of the matrix for that of the determinant. If then so important, why was the idea of the matrix let slip? Of course the writers on this subject had it to commence with. One cannot even define a determinant without the idea of a matrix. The simple fact is that in general the writers on this subject have especially developed those ideas which are naturally expressed in simple algebra, and have postponed or slurred over or omitted altogether those ideas which find their natural expression in multiple algebra. But in this subject

the latter happen to be the fundamental ideas, and those which ought to direct the whole course of thought.

I have taken a very simple illustration, perhaps the very first theorem which meets the student after those immediately connected with the introductory definitions, both because the simplest illustration is really the best, and because I am here most at home. But the principles of multiple algebra seem to me to shed a flood of light into every corner of the subjects usually treated under the title of determinants, the subject gaining as much in breadth from the new notions as in simplicity from the new notations; and in the more intricate subjects of invariants, covariants, etc., I believe that the principles of multiple algebra are ready to perform an equal service. Certainly they make many things seem very simple to me which I should otherwise find difficult of comprehension.

Let us turn to geometry.

If we were asked to characterize in a single word our modern geometry, we would perhaps say that it is a geometry of position. Now position is essentially a multiple quantity, or, if you prefer, is naturally represented in algebra by a multiple quantity. And the growth in this century of the so-called synthetic as opposed to analytical geometry seems due to the fact that by the ordinary analysis geometers could not easily express, except in a cumbersome and unnatural manner, the sort of relations in which they were particularly interested. With the introduction of the notations of multiple algebra, this difficulty falls away, and with it the opposition between synthetic and analytical geometry.

It is, however, interesting and very instructive to observe how the ingenuity of mathematicians has often triumphed over the limitations of ordinary algebra. A conspicuous example and one of the simplest is seen in the *Mécanique Analytique*, where the author, by the use of what are sometimes called indeterminate equations, is able to write in one equation the equivalent of an indefinite number. Thus the equation

$$X\,dx+Y\,dy+Z\,dz=0,$$

by the indeterminateness of the values of dx, dy, dz, is made equivalent to the three equations

$$X=0,\quad Y=0,\quad Z=0.$$

It is instructive to compare this with

$$Xi+Yj+Zk=0,$$

which is the form that Hamilton or Grassmann would have used. The use of this analytical artifice, if such it can be called, runs all through the work and is fairly characteristic of it.

Again, the introduction of the potential in the theory of gravity, or

electricity, or magnetism, gives us a scalar quantity instead of a vector as the subject of study; and in mechanics generally the use of the force-function substitutes a simple quantity for a complex. This method is in reality not different from that just mentioned, since Lagrange's indeterminate equation expresses, at least in its origin, the variation of the force-function. It is indeed the real beauty of Lagrange's method that it is not so much an analytical artifice, as the natural development of the subject.

In modern analytical geometry we find methods in use which are exceedingly ingenious, and give forms curiously like those of multiple algebra, but which, at least if logically carried out very far, are excessively artificial, and that for the expression of the simplest things. The simplest conceptions of the geometry of three dimensions are points and planes, and the simplest relation between these is that a point lies in a plane. Let us see how these notions have been handled by means of ordinary algebra, and by multiple algebra. It will illustrate the characteristic difference of the methods, perhaps as well as the reading of an elaborate treatise.

In multiple algebra a point is designated by a single letter, just as it is in what is called synthetic geometry, and as it generally is by the ordinary analyst when he is not writing equations. But in his equations, instead of a single letter the analyst introduces several letters (coordinates) to represent the point.

A plane may be represented in multiple algebra as in synthetic geometry by a single letter; in the ordinary algebra it is sometimes represented by three coordinates, for which it is most convenient to take the reciprocals of the segments cut off by the plane on three axes. But the modern analyst has a more ingenious method of representing the plane. He observes that the equation of the plane may be written

$$\xi x+\eta y+\zeta z=1, \tag{1}$$

where ξ, η, ζ are the reciprocals of the segments, and x, y, z are the coordinates of any point in the plane. Now if we set

$$p=\xi x+\eta y+\zeta z, \tag{2}$$

this letter will represent an expression which represents the plane. In fact, we may say that p implicitly contains ξ, η, and ζ, which are the coordinates of the plane. We may therefore speak of the plane p, and for many purposes can introduce the letter p into our equations instead of ξ, η, ζ. For example, the equation

$$p'''=\frac{p'+p''}{2} \tag{3}$$

is equivalent to the three equations

$$\xi'''=\frac{\xi'+\xi''}{2}, \qquad \eta'''=\frac{\eta'+\eta''}{2}, \qquad \zeta'''=\frac{\zeta'+\zeta''}{2}. \tag{4}$$

It is to be noticed that on account of the indeterminateness of the x, y, and z, this method, regarded as an analytical artifice, is identical with that of Lagrange, also that in multiple algebra we should have an equation of precisely the same form as (3) to express the same relation between the planes, but that the equation would be explained to the student in a totally different manner. This we shall see more particularly hereafter.

It is curious that we have thus a simpler notation for a plane than for a point. This however may be reversed. If we commence with the notion of the coordinates of a plane, ξ, η, ζ, the equation of a point (i.e., the equation between ξ, η, ζ which will hold for every plane passing through the point) will be

$$x\xi+y\eta+z\zeta=1, \tag{5}$$

where x, y, z are the coordinates of the point. Now if we set

$$q=x\xi+y\eta+z\zeta, \tag{6}$$

we may regard the single letter q as representing the point, and use it, in many cases, instead of the coordinates x, y, z, which indeed it implicitly contains. Thus we may write

$$q'''=\frac{q'+q''}{2} \tag{7}$$

for the three equations

$$x'''=\frac{x'+x''}{2}, \quad y'''=\frac{y'+y''}{2}, \quad z'''=\frac{z'+z''}{2}. \tag{8}$$

Here, by an analytical artifice, we come to equations identical in form and meaning with those used by Hamilton, Grassmann, and even by Möbius in 1827. But the explanations of the formulæ would differ widely. The methods of the founders of multiple algebra are characterized by a bold simplicity, that of the modern geometry by a somewhat bewildering ingenuity. That p and q represent the same expression (in one case x, y, z, and in the other ξ, η, ζ being indeterminate) is a circumstance which may easily become perplexing. I am not quite certain that it would be convenient to use both of these abridged notations at the same time. In fact, if the geometer using these methods were asked to express by an equation in p and q that the point q lies in the plane p, he might find himself somewhat entangled in the meshes of his own ingenuity, and need some new artifice to extricate himself. I do not mean that his genius might not possibly be equal to the occasion, but I do mean very seriously that it is a vicious method which requires any ingenuity or any artifice to express so simple a relation.

If we use the methods of multiple algebra which are most comparable to those just described, a point is naturally represented by a vector (ρ) drawn to it from the origin, a plane by a vector (σ) drawn

from the origin perpendicularly toward the plane and in length equal to the reciprocal of the distance of the plane from the origin. The equation

$$\sigma''' = \frac{\sigma' + \sigma''}{2} \tag{9}$$

will have precisely the same meaning as equation (3), and

$$\rho''' = \frac{\rho' + \rho''}{2} \tag{10}$$

will have precisely the same meaning as equation (7), viz., that the point ρ''' is in the middle between ρ' and ρ''. That the point ρ lies in the plane σ is expressed by equating to unity the product of ρ and σ called by Grassmann internal, or by Hamilton called the scalar part of the product taken negatively. By whatever name called, the quantity in question is the product of the lengths of the vectors and the cosine of the included angle. It is of course immaterial what particular sign we use to express this product, as whether we write

$$\rho.\sigma = 1, \qquad \text{or} \qquad S\rho\sigma = -1. \tag{11}$$

I should myself prefer the simplest possible sign for so simple a relation. It may be observed that ρ and σ may be expressed as the geometrical sum of their components parallel to a set of perpendicular axes, viz.,

$$\rho = xi + yj + zk, \qquad \sigma = \xi i + \eta j + \zeta k. \tag{12}$$

By substitution of these values, equation (11) becomes by the laws of this kind of multiplication

$$x\xi + y\eta + z\zeta = 1. \tag{13}$$

My object in going over these elementary matters is to call attention to the very roundabout way in which the ordinary analysis makes out to represent a point or a plane by a single letter, as distinguished from the directness and simplicity of the notations of multiple algebra, and also to the fact that the representations of points and planes by single letters in the ordinary analysis are not, when obtained, as amenable to analytical treatment as are the notations of multiple algebra.

I have compared that form of the ordinary analysis which relates to Cartesian axes with a vector analysis. But the case is essentially the same if we compare the form of ordinary analysis which relates to a fundamental tetrahedron with Grassmann's geometrical analysis, founded on the point as the elementary quantity.

In the method of ordinary analysis a point is represented by four coordinates, of which each represents the distance of the point from a plane of the tetrahedron divided by the distance of the opposite vertex from the same plane. The equation of a plane may be put in the form

$$\xi x + \eta y + \zeta z + \omega w = 0, \tag{14}$$

where ξ, η, ζ, ω are the distances of the plane from the four points, and x, y, z, w are the coordinates of any point in the plane. Here we may set

$$p = \xi x + \eta y + \zeta z + \omega w, \tag{15}$$

and say that p represents the plane. To some extent we can introduce this letter into equations instead of ξ, η, ζ, ω. Thus the equation

$$lp' + mp'' + np''' = 0 \tag{16}$$

(which denotes that the planes p', p'', p''', meet in a common line, making angles of which the sines are proportional to l, m, and n) is equivalent to the four equations

$$l\xi' + m\xi'' + n\xi''' = 0, \qquad l\eta' + m\eta'' + n\eta''' = 0, \quad \text{etc.} \tag{17}$$

Again, we may regard ξ, η, ζ, ω as the coordinates of a plane. The equation of a point will then be

$$x\xi + y\eta + z\zeta + w\omega = 0. \tag{18}$$

If we set

$$q = x\xi + y\eta + z\zeta + w\omega, \tag{19}$$

we may say that q represents the point. The equation

$$q''' = \frac{q' + q''}{2}, \tag{20}$$

which indicates that the point q''' bisects the line between q' and q'', is equivalent to the four equations

$$\xi' = \frac{\xi'' + \xi'''}{2}, \qquad \eta' = \frac{\eta'' + \eta'''}{2}, \quad \text{etc.} \tag{21}$$

To express that the point q lies in the plane p does not seem easy, without going back to the use of coordinates.

The form of multiple algebra which is to be compared to this is the geometrical algebra of Möbius and Grassmann, in which points without reference to any origin are represented by single letters, say by Italic capitals, and planes may also be represented by single letters, say by Greek capitals. An equation like

$$Q''' = \frac{Q' + Q''}{2}, \tag{22}$$

has exactly the same meaning as equation (20) of ordinary algebra. So

$$l\Pi' + m\Pi'' + n\Pi''' = 0 \tag{23}$$

has precisely the same meaning as equation (16) of ordinary algebra. That the point Q lies in the plane Π is expressed by equating to zero the product of Q and Π which is called by Grassmann external and which might be defined as the distance of the point from the plane. We may write this

$$Q \times \Pi = 0. \tag{24}$$

To show that so simple an expression is really amenable to analytical treatment, I observe that Q may be expressed in terms of any four points (not in the same plane) on the barycentric principle explained above, viz.,

$$Q = xA + yB + zC + wD, \tag{25}$$

and Π may be expressed in terms of combinatorial products of A, B, C, and D, viz.,

$$\Pi = \xi B \times C \times D + \eta C \times A \times D + \zeta D \times A \times B + \omega A \times C \times B, \tag{26}$$

and by these substitutions, by the laws of the combinatorial product to be mentioned hereafter, equation (24) is transformed into

$$w\omega + x\xi + y\eta + z\zeta = 0, \tag{27}$$

which is identical with the formula of ordinary analysis.*

I have gone at length into this very simple point in order to illustrate the fact, which I think is a general one, that the modern geometry is not only tending to results which are appropriately expressed in multiple algebra, but that it is actually striving to clothe itself in forms which are remarkably similar to the notations of multiple algebra, only less simple and general and far less amenable to analytical treatment, and therefore, that a certain logical necessity calls for throwing off the yoke under which analytical geometry has so long labored. And lest this should seem to be the utterance of an uninformed enthusiasm, or the echoing of the possibly exaggerated claims of the devotees of a particular branch of mathematical study, I will quote a sentence from Clebsch and one from Clifford, relating to the past and to the future of multiple algebra. The former in his eulogy on Plücker,† in 1871, speaking of recent advances in geometry, says that "in a certain sense the coordinates of a straight line, and in general a great part of the fundamental conceptions of the newer algebra, are contained in the *Ausdehnungslehre* of 1844," and Clifford ‡ in the last year of his life, speaking of the *Ausdehnungslehre*, with which he had but recently become acquainted, expresses "his profound admiration of that extraordinary work, and his conviction that its principles will exercise a vast influence upon the future of mathematical science."

Another subject in which we find a tendency toward the forms and methods of multiple algebra, is the calculus of operations. Our ordinary analysis introduces operators, and the successive operations A and B may be equivalent to the operation C. To express this in an equation we may write

$$BA(x) = C(x),$$

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* The letters ξ, η, ζ, ω, here denote the distances of the plane Π from the points A, B, C, D, divided by six times the volume of the tetrahedron, A, B, C, D. The letters x, y, z, w, denote the tetrahedral coordinates as above.

† *Gött. Abhandl.*, vol. xvi, p. 28. ‡ *Amer. Journ. Math.*, vol. i, p. 350.

where x is any quantity or function. We may also have occasion to write $$A(x)+B(x)=D(x), \quad \text{or} \quad (A+B)(x)=D(x).$$

But it is almost impossible to resist the tendency to express these relations in the form $$BA=C,$$ $$A+B=D,$$

in which the operators appear in a sense as quantities, i.e., as subjects of functional operation. Now since these operators are often of such nature that they cannot be perfectly specified by a single numerical quantity, when we treat them as quantities they must be regarded as multiple quantities. In this way certain formulæ which essentially belong to multiple algebra get a precarious footing where they are only allowed because they are regarded as abridged notations for equations in ordinary algebra. Yet the logical development of such notations would lead a good way in multiple algebra, and doubtless many investigators have entered the field from this side.

One might also notice, to show how the ordinary algebra is becoming saturated with the notions and notations which seem destined to turn it into a multiple algebra, the notation so common in the higher algebra $$(a, b, c)(x, y, z)$$

for $$ax+by+cz.$$

This is evidently the same as Grassmann's internal product of the multiple quantities (a, b, c) and (x, y, z), or, in the language of quaternions, the scalar part, taken negatively, of the product of the vectors of which a, b, c and x, y, z are the components. A similar correspondence with Grassmann's methods might, I think, be shown in such notations as, for example,

$$(a, b, c, d)(x, y)^3.$$

The free admission of such notations is doubtless due to the fact that they are regarded simply as abridged notations.

The author of the celebrated "Memoir on the Theory of Matrices" goes much farther than this in his use of the forms of multiple algebra. Thus he writes explicitly one equation to stand for several, without the use of any of the analytical artifices which have been mentioned. This work has indeed, as we have seen, been characterized as marking the commencement of multiple algebra,—a view to which we can only take exception as not doing justice to earlier writers.

But the significance of this memoir with regard to the point which I am now considering is that it shows that the chasm so marked in the second quarter of this century is destined to be closed up. Notions and notations for which a Cayley is sponsor will not be

excluded from good society among mathematicians. And if we admit as suitable the notations used in this memoir (where it is noticeable that the author rather avoids multiple algebra, and only uses it very sparingly), we shall logically be brought to use a great deal more. For example, if it is a good thing to write in our equations a single letter to represent a matrix of n^2 numerical quantities, why not use a single letter to represent the n quantities operated upon, as Grassmann and Hamilton have done? Logical consistency seems to demand it. And if we may use the sign)(to denote an operation by which two sets of quantities are combined to form a third set, as is the case in this memoir, why not use other signs to denote other functional operations of which the result is a multiple quantity? If it be conceded that this is the proper method to follow where simplicity of conception, or brevity of expression, or ease of transformation is served thereby, our algebra will become in large part a multiple algebra.

We have considered the subject a good while from the outside; we have glanced at the principal events in the history of multiple algebra; we have seen how the course of modern thought seems to demand its aid, how it is actually leaning toward it, and beginning to adopt its methods. It may be worth while to direct our attention more critically to multiple algebra itself, and inquire into its essential character and its most important principles.

I do not know that anything useful or interesting, which relates to multiple quantity, and can be symbolically expressed, falls outside of the domain of multiple algebra. But if it is asked, what notions are to be regarded as fundamental, we must answer, here as elsewhere, those which are most simple and fruitful. Unquestionably, no relations are more so than those which are known by the names of addition and multiplication.

Perhaps I should here notice the essentially different manner in which the multiplication of multiple quantities has been viewed by different writers. Some, as Hamilton, or De Morgan, or Peirce, speak of the product of two multiple quantities, as if only one product could exist, at least in the same algebra. Others, as Grassmann, speak of various kinds of products for the same multiple quantities. Thus Hamilton seems for many years to have agitated the question, what he should regard as the product of each pair of a set of triplets, or in the geometrical application of the subject, what he should regard as the product of each pair of a system of perpendicular directed lines.* Grassmann asks, What products, i.e., what distributive functions of the multiple quantities, are most important?

* *Phil. Mag.* (3), vol. xxv, p. 490; *North British Review*, vol. xlv, (1866), p. 57.

It may be that in some cases the fact that only one kind of product is known in ordinary algebra has led those to whom the problem presented itself in the form of finding a new algebra to adopt this characteristic derived from the old. Perhaps the reason lies deeper in a distinction like that in arithmetic between concrete and abstract numbers or quantities. The multiple quantities corresponding to concrete quantities such as ten apples or three miles are evidently such combinations as ten apples + seven oranges, three miles northward + five miles eastward, or six miles in a direction fifty degrees east of north. Such are the fundamental multiple quantities from Grassmann's point of view. But if we ask what it is in multiple algebra which corresponds to an abstract number like twelve, which is essentially an operator, which changes one mile into twelve miles, and $1,000 into $12,000, the most general answer would evidently be: an operator which will work such changes as, for example, that of ten apples + seven oranges into fifty apples + 100 oranges, or that of one vector into another.

Now an operator has, of course, one characteristic relation, viz., its relation to the operand. This needs no especial definition, since it is contained in the definition of the operator. If the operation is distributive, it may not inappropriately be called multiplication, and the result is *par excellence* the product of the operator and operand. The sum of operators *quâ* operators, is an operator which gives for the product the sum of the products given by the operators to be added. The product of two operators is an operator which is equivalent to the successive operations of the factors. This multiplication is necessarily associative, and its definition is not really different from that of the operators themselves. And here I may observe that Professor C. S. Peirce has shown that his father's associative algebras may be regarded as operational and matricular.*

Now the calculus of distributive operators is a subject of great extent and importance, but Grassmann's view is the more comprehensive, since it embraces the other with something besides. For every quantitative operator may be regarded as a quantity, i.e., as the subject of mathematical operation, but every quantity cannot be regarded as an operator; precisely as in grammar every verb may be taken as substantive, as in the infinitive, while every substantive does not give us a verb.

Grassmann's view seems also the most practical and convenient. For we often use many functions of the same pair of multiple quantities, which are distributive with respect to both, and we need some simple designation to indicate a property of such fundamental

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* *Amer. Journ. Math.*, vol. iv, p. 221.

importance in the algebra of such functions, and no advantage appears in singling out a particular function to be alone called the product. Even in quaternions, where Hamilton speaks of only one product of two vectors (regarding it as a special case of the product of quaternions, i.e., of operators), he nevertheless comes to use the scalar part of this product and the vector part separately. Now the distributive law is satisfied by each of these, which therefore may conveniently be called products. In this sense we have three kinds of products of vectors in Hamilton's analysis.

Let us then adopt the more general view of multiplication, and call any function of two or more multiple quantities, which is distributive with respect to all, a product, with only this limitation, that when one of the factors is simply an ordinary algebraic quantity, its effect is to be taken in the ordinary sense.

It is to be observed that this definition of multiplication implies that we have an addition both of the kind of quantity to which the product belongs, and of the kinds to which the factors belong. Of course, these must be subject to the general formal laws of addition. I do not know that it is necessary for the purposes of a general discussion to stop to define these operations more particularly, either on their own account or to complete the definition of multiplication. Algebra, as a formal science, may rest on a purely formal foundation. To take our illustration again from mechanics, we may say that if a man is inventing a particular machine,—a sewing machine, a reaper,—nothing is more important than that he should have a precise idea of the operation which his machine is to perform, yet when he is treating the general principles of mechanics he may discuss the lever, or the form of the teeth of wheels which will transmit uniform motion, without inquiring the purpose to which the apparatus is to be applied; and in like manner that if we were forming a particular algebra,—a geometrical algebra, a mechanical algebra, an algebra for the theory of elimination and substitution, an algebra for the study of quantics,—we should commence by asking, What are the multiple quantities, or sets of quantities, which we have to consider? What are the additive relations between them? What are the multiplicative relations between them? etc., forming a perfectly defined and complete idea of these relations as we go along; but in the development of a general algebra no such definiteness of conception is requisite. Given only the purely formal law of the distributive character of multiplication,—this is sufficient for the foundation of a science. Nor will such a science be merely a pastime for an ingenious mind. It will serve a thousand purposes in the formation of particular algebras. Perhaps we shall find that in the most important

cases the particular algebra is little more than an application or interpretation of the general.

Grassmann observes that any kind of multiplication of n-fold quantities is characterized by the relations which hold between the products of n independent units. In certain kinds of multiplication these characteristic relations will hold true of the products of any of the quantities.

Thus if the value of a product is independent of the order of the factors when these belong to the system of units, it will always be independent of the order of the factors. The kind of multiplication characterized by this relation and no other between the products is called by Grassmann *algebraic*, because its rules coincide with those of ordinary algebra. It is to be observed, however, that it gives rise to multiple quantities of higher orders. If n independent units are required to express the original quantities, $n\frac{n+1}{2}$ units will be required for the products of two factors, $n\frac{(n+1)(n+2)}{2.3}$ for the products of three factors, etc.

Again, if the value of a product of factors belonging to a system of units is multiplied by -1 when two factors change places, the same will be true of the product of any factors obtained by addition of the units. The kind of multiplication characterized by this relation and no other is called by Grassmann *external* or *combinatorial*. For our present purpose we may denote it by the sign $\times$. It gives rise to multiple quantities of higher orders, $n\frac{n-1}{2}$ units being required to express the products of two factors, $n\frac{(n-1)(n-2)}{2.3}$ units for products of three factors, etc. All products of more than n factors are zero. The products of n factors may be expressed by a single unit, viz., the product of the n original units taken in a specified order, which is generally set equal to 1. The products of $n-1$ factors are expressed in terms of n units, those of $n-2$ factors in terms of $n\frac{n-1}{2}$ units, etc. This kind of multiplication is associative, like the algebraic.

Grassmann observes, with respect to binary products, that these two kinds of multiplication are the only kinds characterized by laws which are the same for any factors as for particular units, except indeed that characterized by no special laws, and that for which all products are zero.* The last we may evidently reject as nugatory. That for which there are no special laws, i.e., in which no equations

* Crelle's *Journ. f. Math.*, vol. xlix, p. 138.

subsist between the products of a system of independent units, is also rejected by Grassmann, as not appearing to afford important applications. I shall, however, have occasion to speak of it, and shall call it the indeterminate product. In this kind of multiplication, n^2 units are required to express the products of two factors, and n^3 units for products of three factors, etc. It evidently may be regarded as associative.

Another very important kind of multiplication is that called by Grassmann *internal*. In the form in which I shall give it, which is less general than Grassmann's, it is in one respect the most simple of all, since its only result is a numerical quantity. It is essentially binary and characterized by laws of the form

$$i.i=1, \qquad j.j=1, \qquad k.k=1, \quad \text{etc.},$$
$$i.j=0, \qquad j.i=0, \qquad \text{etc.},$$

where i, j, k, etc., represent a system of independent units. I use the dot as significant of this kind of multiplication.

Grassmann derives this kind of multiplication from the combinatorial by the following process. He defines the complement (Ergänzung) of a unit as the combinatorial product of all the other units, taken with such a sign that the combinatorial product of the unit and its complement shall be positive. The combinatorial product of a unit and its complement is therefore unity, and that of a unit and the complement of any other unit is zero. The internal product of two units is the combinatorial product of the first and the complement of the second.

It is important to observe that any scalar product of two factors of the same kind of multiple quantities, which is positive when the factors are identical, may be regarded as an internal product, i.e., we may always find such a system of units, that the characteristic equations of the product will reduce to the above form. The nature of the subject may afford a definition of the product independent of any reference to a system of units. Such a definition will then have obvious advantages. An important case of this kind occurs in geometry in that product of two vectors which is obtained by multiplying the products of their lengths by the cosine of the angle which they include. This is an internal product in Grassmann's sense.

Let us now return to the indeterminate product, which I am inclined to regard as the most important of all, since we may derive from it the algebraic and the combinatorial. For this end we will prefix Σ to an indeterminate product to denote the sum of all the terms obtained by taking the factors in every possible order. Then,

$$\Sigma\alpha|\beta|\gamma,$$

for instance, where the vertical line is used to denote the

indeterminate product,* is a distributive function of α, β, and γ. It is evidently not affected by changing the order of the letters. It is, therefore, an algebraic product in the sense in which the term has been defined.

So, again, if we prefix $\Sigma\pm$ to an indeterminate product to denote the sum of all terms obtained by giving the factors every possible order, those terms being taken negatively which are obtained by an odd number of simple permutations,

$$\Sigma\pm\alpha|\beta|\gamma,$$

for instance, will be a distributive function of α, β, γ, which is multiplied by -1 when two of these letters change places. It will therefore be a combinatorial product.

It is a characteristic and very important property of an indeterminate product that every product of all its factors with any other quantities is also a product of the indeterminate product and the other quantities. We need not stop for a formal proof of this proposition, which indeed is an immediate consequence of the definitions of the terms.

These considerations bring us naturally to what Grassmann calls *regressive multiplication*, which I will first illustrate by a very simple example. If n, the degree of multiplicity of our original quantities, is 4, the combinatorial product of $\alpha\times\beta\times\gamma$ and $\delta\times\epsilon$, viz.,

$$\alpha\times\beta\times\gamma\times\delta\times\epsilon,$$

is necessarily zero, since the number of factors exceeds four. But if for $\delta\times\epsilon$ we set its equivalent

$$\delta|\epsilon-\epsilon|\delta,$$

we may multiply the first factor in each of these indeterminate products combinatorially by $\alpha\times\beta\times\gamma$, and prefix the result, which is a numerical quantity, as coefficient to the second factor. This will give

$$(\alpha\times\beta\times\gamma\times\delta)\epsilon-(\alpha\times\beta\times\gamma\times\epsilon)\delta.$$

Now, the first term of this expression is a product of $\alpha\times\beta\times\gamma$, δ, and ϵ, and therefore, by the principle just stated, a product of $\alpha\times\beta\times\gamma$ and $\delta|\epsilon$. The second term is a similar product of $\alpha\times\beta\times\gamma$ and $\epsilon|\delta$. Therefore the whole expression is a product of $\alpha\times\beta\times\gamma$ and $\delta|\epsilon-\epsilon|\delta$, that is, of $\alpha\times\beta\times\gamma$ and $\delta\times\epsilon$. That is, except in sign, what Grassmann calls the *regressive product* of $\alpha\times\beta\times\gamma$ and $\delta\times\epsilon$.

To generalize this process, we first observe that an expression of the form

$$\Sigma\pm\alpha\times\beta|\gamma\times\delta,$$

in which each term is an indeterminate product of two combinatorial products, and in which $\Sigma\pm$ denotes the sum of all terms obtained by

* This notation must not be confounded with Grassmann's use of the vertical line.

putting every different pair of the letters before the dividing line, the negative sign being used for any terms which may be obtained by an odd number of simple permutations of the letters,—in other words, the expression

$$\alpha\times\beta|\gamma\times\delta-\alpha\times\gamma|\beta\times\delta-\alpha\times\delta|\gamma\times\beta+\beta\times\gamma|\alpha\times\delta$$
$$-\beta\times\delta|\alpha\times\gamma+\gamma\times\delta|\alpha\times\beta,$$

is a distributive function of α, β, γ, and δ, which is multiplied by -1 when two of these letters change places, and may, therefore, be regarded as equivalent to the combinatorial product $\alpha\times\beta\times\gamma\times\delta$. Now, if $n=5$, the combinatorial product of

$$\rho\times\sigma\times\tau \qquad \text{and} \qquad \alpha\times\beta\times\gamma\times\delta$$

is zero. But if we multiply the first member of each of the above indeterminate products by $\rho\times\sigma\times\tau$, and prefix the result as coefficient to the second member, we obtain

$$(\rho\times\sigma\times\tau\times\alpha\times\beta)\gamma\times\delta-(\rho\times\sigma\times\tau\times\alpha\times\gamma)\beta\times\delta+\text{etc.},$$

which is what Grassmann calls the regressive product of $\rho\times\sigma\times\tau$ and $\alpha\times\beta\times\gamma\times\delta$. It is easy to see that the principle may be extended so as to give a regressive product in any case in which the total number of factors of two combinatorial products is greater than n. Also, that we might form a regressive product by treating the first of the given combinatorials as we have treated the second. It may easily be shown that this would give the same result, except in some cases with a difference of sign. To avoid this inconvenience, we may make the rule, that whenever in the substitution of a sum of indeterminate products for a combinatorial, both factors of the indeterminate products are of odd degree, we change the sign of the whole expression. With this understanding, the results which we obtain will be identical with Grassmann's regressive product. The propriety of the name consists in the fact that the product is of less degree than either of the factors. For the contrary reason, the ordinary external or combinatorial multiplication is sometimes called by Grassmann *progressive*.

Regressive multiplication is associative and exhibits a very remarkable analogy with the progressive. This analogy I have not time here to develop, but will only remark that in this analogy lies in its most general form that celebrated *principle of duality*, which appears in various forms in geometry and certain branches of analysis.

To fix our ideas, I may observe that in geometry the progressive multiplication of points gives successively lines, planes and volumes; the regressive multiplication of planes gives successively lines, points and scalar quantities.

The indeterminate product affords a natural key to the subject of matrices. In fact, a sum of indeterminate products of the second

degree represents n^2 scalars, which constitute an ordinary or quadratic matrix; a sum of indeterminate products of the third degree represents n^3 scalars, which constitute a cubic matrix, etc. I shall confine myself to the simplest and most important case, that of quadratic matrices.

An expression of the form

$$\alpha(\lambda.\rho)$$

being a product of α, λ, and ρ, may be regarded as a product of $\alpha|\lambda$ and ρ, by a principle already stated. Now if Φ denotes a sum of indeterminate products, of second degree, say $\alpha|\lambda+\beta|\mu+\text{etc.}$, we may write

$$\Phi.\rho$$

for $$\alpha(\lambda.\rho)+\beta(\mu.\rho)+\text{etc.}$$

This is like ρ, a quantity of the first degree, and it is a homogeneous linear function of ρ. It is easy to see that the most general form of such a function may be expressed in this way. An equation like

$$\sigma=\Phi.\rho$$

represents n equations in ordinary algebra, in which n variables are expressed as linear functions of n others by means of n^2 coefficients.

The internal product of two indeterminate products may be defined by the equation $$(\alpha|\beta).(\gamma|\delta)=(\beta.\gamma)\,\alpha|\delta.$$

This defines the internal product of matrices, as

$$\Psi.\Phi.$$

This product evidently gives a matrix, the operation of which is equivalent to the successive operations of Φ and Ψ; i.e.,

$$(\Psi.\Phi).\rho=\Psi.(\Phi.\rho).$$

We may express this a little more generally by saying that internal multiplication is associative when performed on a series of matrices, or on such a series terminated by a quantity of the first degree.

Another kind of multiplication of binary indeterminate products is that in which the preceding factors are multiplied combinatorially, and also the following. It may be defined by the equation

$$(\alpha|\lambda)^{\times}_{\times}(\beta|\mu)^{\times}_{\times}(\gamma|\nu)=\alpha\times\beta\times\gamma|\lambda\times\mu\times\nu.$$

This defines a multiplication of matrices denoted by the same symbol, as $$\Phi^{\times}_{\times}\Psi^{\times}_{\times}\Omega, \qquad \Phi^{\times}_{\times}\Psi^{\times}_{\times}\Omega^{\times}_{\times}\Theta.$$

This multiplication, which is associative and commutative, is of great importance in the theory of determinants. In fact,

$$\frac{1}{\lfloor n}\Phi^{\times n}_{\times}$$

is the determinant of the matrix Φ. A lower power, as the m^{th}, with the divisor $n(n-1)\dots(n-m+1)$ would express as multiple quantity all the subdeterminants of order m.*

It is evident that by the combination of the operations of indeterminate, algebraic, and combinatorial multiplication we obtain multiple quantities of a more complicated nature than by the use of only one of these kinds of multiplication. The indeterminate product of combinatorial products we have already mentioned. The combinatorial product of algebraic products, and the indeterminate product of algebraic products, are also of great importance, especially in the theory of quantics. These three multiplications, with the internal, especially in connection with the general property of the indeterminate product given above, and the derivation of the algebraic and combinatorial products from the indeterminate, which affords a generalization of that property, give rise to a great wealth of multiplicative relations between these multiple quantities. I say "*wealth* of multiplicative relations" designedly, for there is hardly any kind of relations between things which are the objects of mathematical study, which add so much to the resources of the student as those which we call multiplicative, except perhaps the simpler class which we call additive, and which are presupposed in the multiplicative. This is a truth quite independent of our using any of the notations of multiple algebra, although a suitable notation for such relations will of course increase their value.

Perhaps, before closing, I ought to say a few words on the applications of multiple algebra.

First of all, geometry, and the geometrical sciences which treat of things having position in space, kinematics, mechanics, astronomy physics, crystallography, seem to demand a method of this kind, for position in space is essentially a multiple quantity and can only be represented by simple quantities in an arbitrary and cumbersome manner. For this reason, and because our spatial intuitions are more developed than those of any other class of mathematical relations, these subjects are especially adapted to introduce the student to the methods of multiple algebra. Here, Nature herself takes us by the hand and leads us along by easy steps, as a mother teaches her child to walk. In the contemplation of such subjects, Möbius, Hamilton,

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* Quadratic matrices may also be represented by a sum of indeterminate products of a quantity of the first degree with a combinatorial product of $(n-1)$st degree, as, for example, when $n=4$, by a sum of products of the form

$$\alpha|\beta\times\gamma\times\delta.$$

The theory of such matrices is almost identical with that of those of the other form, except that the external multiplication takes the place of the internal, in the multiplication of the matrices with each other and with quantities of the first degree.

and Grassmann formed their algebras, although the philosophical mind of the last was not satisfied until he had produced a system unfettered by any spatial relations. It is probably in connection with some of these subjects that the notions of multiple algebra are most widely disseminated.

Maxwell's *Treatise on Electricity and Magnetism* has done so much to familiarize students of physics with quaternion notations, that it seems impossible that this subject should ever again be entirely divorced from the methods of multiple algebra.

I wish that I could say as much of astronomy. It is, I think, to be regretted, that the oldest of the scientific applications of mathematics, the most dignified, the most conservative, should keep so far aloof from the youngest of mathematical methods; and standing as I do to-day, by some chance, among astronomers, although not of the guild, I cannot but endeavor to improve the opportunity by expressing my conviction of the advantages which astronomers might gain by employing some of the methods of multiple algebra. A very few of the fundamental notions of a vector analysis, the addition of vectors and what quaternionists would call the scalar part and the vector part of the product of two vectors (which may be defined without the notion of the quaternion),—these three notions with some four fundamental properties relating to them are sufficient to reduce enormously the labor of mastering such subjects as the elementary theory of orbits, the determination of an orbit from three observations, the differential equations which are used in determining the best orbit from an indefinite number of observations by the method of least squares, or those which give the perturbations when the elements are treated as variable. In all these subjects the analytical work is greatly simplified, and it is far easier to find the best form for numerical calculation than by the use of the ordinary analysis.

I may here remark that in its geometrical applications multiple algebra will naturally take one of two principal forms, according as vectors or points are taken as elementary quantities, i.e., according as something having magnitude and direction, or something having magnitude and position at a point, is the fundamental conception. These forms of multiple algebra may be distinguished as *vector analysis* and *point analysis*. The former we may call a triple, the latter a quadruple algebra, if we determine the degree of the algebra from the degree of multiplicity of the fundamental conception. The former is included in the latter, since the subtraction of points gives us vectors, and in this way Grassmann's vector analysis is included in his point analysis. Hamilton's system, in which the vector is the fundamental idea, is nevertheless made a quadruple algebra by the addition of ordinary numerical quantities. For practical purposes we

may regard Hamilton's system as equivalent to Grassmann's algebra of vectors. Such practical equivalence is of course consistent with great differences of notation, and of the point of view from which the subject is regarded.

Perhaps I should add a word in regard to the nature of the problems which require a vector analysis, or the more general form of Grassmann's point analysis. The distinction of the problems is very marked, and corresponds precisely to the distinction familiar to all analysts between problems which are suitable for Cartesian coordinates, and those which are suitable for the use of tetrahedral, or, in plane geometry, triangular coordinates. Thus, in mechanics, kinematics, astronomy, physics, or crystallography, Grassmann's point analysis will rarely be wanted. One might teach these subjects for years by a vector analysis, and never perhaps feel the need of any of the notions or notations which are peculiar to the point analysis, precisely as in ordinary algebra one might use the Cartesian coordinates in teaching these subjects, without any occasion for the use of tetrahedral coordinates. I think of one exception, which, however, confirms the rule. The very important theory of forces acting on a rigid body is much better treated by point analysis than by vector analysis, exactly as in ordinary algebra it is much better treated by tetrahedral coordinates than by Cartesian,—I mean for the purpose of the elegant development of general propositions. A sufficient theory for the purposes of numerical calculations can easily enough be given by any method, and the most familiar to the student is for such practical purposes of course the best. On the other hand, the projective properties of bodies, the relations of collinearity, and similar subjects, seem to demand the point analysis for their adequate treatment.

If I have said that the algebra of vectors is contained in the algebra of points, it does not follow that in a certain sense the algebra of points is not deducible from the algebra of vectors. In mathematics, a part often contains the whole. If we represent points by vectors drawn from a common origin, and then develop those relations between such vectors representing points, which are independent of the position of the origin,—by this simple process we may obtain a large part, possibly all, of an algebra of points. In this way the vector analysis may be made to serve very conveniently for many of those subjects which I have mentioned as suitable for point analysis. The vector analysis, thus enlarged, is hardly to be distinguished from a point analysis, but the treatment of the subject in this way has somewhat of a makeshift character, as distinguished from the unity and simplicity of the subject when developed directly from the idea of something situated at a point.

Of those subjects which have no relations to space, the elementary

theory of eliminations and substitutions, including the theory of matrices and determinants, seems to afford the most simple application of multiple algebra. I have already indicated what seems to me the appropriate foundation for the theory of matrices. The method is essentially that which Grassmann has sketched in his first *Ausdehnungslehre* under the name of the *open product* and has developed at length in the second.

In the theory of quantics Grassmann's algebraic product finds an application, the quantic appearing as a sum of algebraic products in Grassmann's sense of the term. As it has been stated that these products are subject to the same laws as the ordinary products of algebra, it may seem that we have here a distinction without an important difference. If the quantics were to be subject to no farther multiplications, except the algebraic in Grassmann's sense, such an objection would be valid. But quantics regarded as sums of algebraic products, in Grassmann's sense, are multiple quantities and subject to a great variety of other multiplications than the algebraic, by which they were formed. Of these the most important are doubtless the combinatorial, the internal, and the indeterminate. The combinatorial and the internal may be applied, not only to the quantic as a whole or to the algebraic products of which it consists, but also to the individual factors in each term, in accordance with the general principle which has been stated with respect to the indeterminate product and which will apply also to the algebraic, since the algebraic may be regarded as a sum of indeterminate products.

In the differential and integral calculus it is often advantageous to regard as multiple quantities various sets of variables, especially the independent variables, or those which may be taken as such. It is often convenient to represent in the form of a single differential coefficient, as

$$\frac{d\tau}{d\rho},$$

a block or matrix of ordinary differential coefficients. In this expression, ρ may be a multiple quantity representing say n independent variables, and τ another representing perhaps the same number of dependent variables. Then $d\rho$ represents the n differentials of the former, and $d\tau$ the n differentials of the latter. The whole expression represents an operator which turns $d\rho$ into $d\tau$, so that we may write identically

$$d\tau = \frac{d\tau}{d\rho} d\rho.$$

Here we see a matrix of n^2 differential coefficients represented by a quotient. This conception is due to Grassmann, as well as the representation of the matrix by a sum of products, which we have

already considered. It is to be observed that these multiple differential coefficients are subject to algebraic laws very similar to those which relate to ordinary differential coefficients when there is a single independent variable, e.g.,

$$\frac{d\sigma}{d\tau}\frac{d\tau}{d\rho}=\frac{d\sigma}{d\rho},$$

$$\frac{d\rho}{d\tau}\frac{d\tau}{d\rho}=1.$$

In the integral calculus, the transformation of multiple integrals by change of variables is made very simple and clear by the methods of multiple algebra.

In the geometrical applications of the calculus, there is a certain class of theorems, of which Green's and Poisson's are the most notable examples, which seem to have been first noticed in connection with certain physical theories, especially those of electricity and magnetism, and which have only recently begun to find their way into treatises on the calculus. These not only find simplicity of expression and demonstration in the infinitesimal calculus of multiple quantities, but also their natural position, which they hardly seem to find in the ordinary treatises.

But I do not so much desire to call your attention to the diversity of the applications of multiple algebra, as to the simplicity and unity of its principles. The student of multiple algebra suddenly finds himself freed from various restrictions to which he has been accustomed. To many, doubtless, this liberty seems like an invitation to license. Here is a boundless field in which caprice may riot. It is not strange if some look with distrust for the result of such an experiment. But the farther we advance, the more evident it becomes that this too is a realm subject to law. The more we study the subject, the more we find all that is most useful and beautiful attaching itself to a few central principles. We begin by studying *multiple algebras;* we end, I think, by studying

V.

ON THE DETERMINATION OF ELLIPTIC ORBITS FROM THREE COMPLETE OBSERVATIONS.

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THE determination of an orbit from three complete observations by the solution of the equations which represent elliptic motion presents so great difficulties in the general case, that in the first solution of the problem we must generally limit ourselves to the case in which the intervals between the observations are not very long. In this case we substitute some comparatively simple relations between the unknown quantities of the problem, which have an approximate validity for short intervals, for the less manageable relations which rigorously subsist between these quantities. A comparison of the approximate solution thus obtained with the exact laws of elliptic motion will always afford the means of a closer approximation, and by a repetition of this process we may arrive at any required degree of accuracy.

It is therefore a problem not without interest—it is, in fact, the natural point of departure in the study of the determination of orbits —to express in a manner combining as far as possible simplicity and accuracy the relations between three positions in an orbit separated by small or moderate intervals. The problem is not entirely determinate, for we may lay the greater stress upon simplicity or upon accuracy; we may seek the most simple relations which are sufficiently accurate to give us any approximation to an orbit, or we may seek the most exact expression of the real relations, which shall not be too complex to be serviceable.

Derivation of the Fundamental Equation.

The following very simple considerations afford a vector equation, not very complex and quite amenable to analytical transformation, which expresses the relations between three positions in an orbit separated by small or moderate intervals, with an accuracy far exceeding that of the approximate relations generally used in the determination of orbits.

If we adopt such a unit of time that the acceleration due to the sun's action is unity at a unit's distance, and denote the vectors* drawn from the sun to the body in its three positions by $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$, and the lengths of these vectors (the heliocentric distances) by r_1, r_2, r_3, the accelerations corresponding to the three positions will be represented by $-\frac{\mathfrak{R}_1}{r_1^3}$, $-\frac{\mathfrak{R}_2}{r_2^3}$, $-\frac{\mathfrak{R}_3}{r_3^3}$. Now the motion between the positions considered may be expressed with a high degree of accuracy by an equation of the form

$$\mathfrak{R}=\mathfrak{A}+t\mathfrak{B}+t^2\mathfrak{C}+t^3\mathfrak{D}+t^4\mathfrak{E},$$

having five vector constants. The actual motion rigorously satisfies six conditions, viz., if we write τ_3 for the interval of time between the

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* Vectors, or *directed quantities*, will be represented in this paper by German capitals. The following notations will be used in connection with them:

The sign = denotes identity in direction as well as length.

The sign + denotes geometrical addition, or what is called composition in mechanics.

The sign − denotes reversal of direction, or composition after reversal.

The notation $\mathfrak{A}.\mathfrak{B}$ denotes the product of the lengths of the vectors and the cosine of the angle which they include. It will be called the *direct product* of $\mathfrak{A}$ and $\mathfrak{B}$. If x, y, z are the rectangular components of $\mathfrak{A}$, and x', y', z' those of $\mathfrak{B}$,

$$\mathfrak{A}.\mathfrak{B}=xx'+yy'+zz'.$$

$\mathfrak{A}.\mathfrak{A}$ may be written $\mathfrak{A}^2$ and called the square of $\mathfrak{A}$.

The notation $\mathfrak{A}\times\mathfrak{B}$ will be used to denote a vector of which the length is the product of the lengths of $\mathfrak{A}$ and $\mathfrak{B}$ and the sine of the angle which they include. Its direction is perpendicular to $\mathfrak{A}$ and $\mathfrak{B}$, and on that side on which a rotation from $\mathfrak{A}$ to $\mathfrak{B}$ appears counter-clockwise. It will be called the *skew product* of $\mathfrak{A}$ and $\mathfrak{B}$. If the rectangular components of $\mathfrak{A}$ and $\mathfrak{B}$ are x, y, z, and x', y', z', those of $\mathfrak{A}\times\mathfrak{B}$ will be

$$yz'-zy', \quad zx'-xz', \quad xy'-yx'.$$

The notation $(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ denotes the volume of the parallelopiped of which three edges are obtained by laying off the vectors $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ from any same point, which volume is to be taken positively or negatively, according as the vector $\mathfrak{C}$ falls on the side of the plane containing $\mathfrak{A}$ and $\mathfrak{B}$, on which a rotation from $\mathfrak{A}$ to $\mathfrak{B}$ appears counter-clockwise, or on the other side. If the rectangular components of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$ are x, y, z; x', y', z'; and x'', y'', z'',

$$(\mathfrak{A}\mathfrak{B}\mathfrak{C})=\begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}.$$

It follows, from the above definitions, that for any vectors $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$

$$\mathfrak{A}.\mathfrak{B}=\mathfrak{B}.\mathfrak{A}, \qquad \mathfrak{A}\times\mathfrak{B}=-\mathfrak{B}\times\mathfrak{A},$$

$$(\mathfrak{A}\mathfrak{B}\mathfrak{C})=(\mathfrak{B}\mathfrak{C}\mathfrak{A})=(\mathfrak{C}\mathfrak{A}\mathfrak{B})=-(\mathfrak{A}\mathfrak{C}\mathfrak{B})=-(\mathfrak{C}\mathfrak{B}\mathfrak{A})=-(\mathfrak{B}\mathfrak{A}\mathfrak{C}),$$

and

$$(\mathfrak{A}\mathfrak{B}\mathfrak{C})=\mathfrak{A}.(\mathfrak{B}\times\mathfrak{C})=\mathfrak{B}.(\mathfrak{C}\times\mathfrak{A})=\mathfrak{C}.(\mathfrak{A}\times\mathfrak{B});$$

also that $\mathfrak{A}.\mathfrak{B}$, $\mathfrak{A}\times\mathfrak{B}$, are distributive functions of $\mathfrak{A}$ and $\mathfrak{B}$, and $(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ a distributive function of $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{C}$, for example, that if $\mathfrak{A}=\mathfrak{L}+\mathfrak{M}$,

$$\mathfrak{A}.\mathfrak{B}=\mathfrak{L}.\mathfrak{B}+\mathfrak{M}.\mathfrak{B}, \qquad \mathfrak{A}\times\mathfrak{B}=\mathfrak{L}\times\mathfrak{B}+\mathfrak{M}\times\mathfrak{B}, \qquad (\mathfrak{A}\mathfrak{B}\mathfrak{C})=(\mathfrak{L}\mathfrak{B}\mathfrak{C})+(\mathfrak{M}\mathfrak{B}\mathfrak{C}),$$

and so for $\mathfrak{B}$ and $\mathfrak{C}$.

The notation $(\mathfrak{A}\mathfrak{B}\mathfrak{C})$ is identical with that of Lagrange in the *Mécanique Analytique*, except that there its use is limited to unit vectors. The signification of $\mathfrak{A}\times\mathfrak{B}$ is closely related to, but not identical with, that of the notation $[r_1 r_2]$ commonly used to denote the double area of a triangle determined by two positions in an orbit.

first and second positions, and τ_1 for that between the second and third, and set $t=0$ for the second position,

for $t=-\tau_3$,

$$\mathfrak{R}=\mathfrak{R}_1, \qquad \frac{d^2\mathfrak{R}}{dt^2}=-\frac{\mathfrak{R}_1}{r_1^3};$$

for $t=0$,

$$\mathfrak{R}=\mathfrak{R}_2, \qquad \frac{d^2\mathfrak{R}}{dt^2}=-\frac{\mathfrak{R}_2}{r_2^3};$$

for $t=\tau_1$,

$$\mathfrak{R}=\mathfrak{R}_3, \qquad \frac{d^2\mathfrak{R}}{dt^2}=-\frac{\mathfrak{R}_3}{r_3^3}.$$

We may therefore write with a high degree of approximation

$$\mathfrak{R}_1=\mathfrak{A}-\tau_3\mathfrak{B}+\tau_3^2\mathfrak{C}-\tau_3^3\mathfrak{D}+\tau_3^4\mathfrak{E}$$
$$\mathfrak{R}_2=\mathfrak{A}$$
$$\mathfrak{R}_3=\mathfrak{A}+\tau_1\mathfrak{B}+\tau_1^2\mathfrak{C}+\tau_1^3\mathfrak{D}-\tau_1^4\mathfrak{E}$$
$$-\frac{\mathfrak{R}_1}{r_1^3}=2\mathfrak{C}-6\tau_3\mathfrak{D}+12\tau_3^2\mathfrak{E}$$
$$-\frac{\mathfrak{R}_2}{r_2^3}=2\mathfrak{C}$$
$$-\frac{\mathfrak{R}_3}{r_3^3}=2\mathfrak{C}+6\tau_1\mathfrak{D}+12\tau_1^2\mathfrak{E}.$$

From these six equations the five constants $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ may be eliminated, leaving a single equation of the form

$$A_1\left(1+\frac{B_1}{r_1^3}\right)\mathfrak{R}_1-\left(1-\frac{B_2}{r_2^3}\right)\mathfrak{R}_2+A_3\left(1+\frac{B_3}{r_3^3}\right)\mathfrak{R}_3=0, \tag{1}$$

where

$$A_1=\frac{\tau_1}{\tau_1+\tau_3}, \quad A_3=\frac{\tau_3}{\tau_1+\tau_3},$$
$$B_1=\tfrac{1}{12}(-\tau_1^2+\tau_1\tau_3+\tau_3^2), \quad B_2=\tfrac{1}{12}(\tau_1^2+3\tau_1\tau_3+\tau_3^2)$$
$$B_3=\tfrac{1}{12}(\tau_1^2+\tau_1\tau_3-\tau_3^2).$$

This we shall call our fundamental equation. In order to discuss its geometrical signification, let us set

$$n_1=A_1\left(1+\frac{B_1}{r_1^3}\right), \quad n_2=\left(1-\frac{B_2}{r_2^3}\right), \quad n_3=A_3\left(1+\frac{B_3}{r_3^3}\right), \tag{2}$$

so that the equation will read

$$n_1\mathfrak{R}_1-n_2\mathfrak{R}_2+n_3\mathfrak{R}_3=0. \tag{3}$$

This expresses that the vector $n_2\mathfrak{R}_2$ is the diagonal of a parallelogram of which $n_1\mathfrak{R}_1$ and $n_3\mathfrak{R}_3$ are sides. If we multiply by $\mathfrak{R}_3$ and by $\mathfrak{R}_1$, *in skew multiplication*, we get

$$n_1\mathfrak{R}_1\times\mathfrak{R}_3-n_2\mathfrak{R}_2\times\mathfrak{R}_3=0, \quad -n_2\mathfrak{R}_1\times\mathfrak{R}_2+n_3\mathfrak{R}_1\times\mathfrak{R}_3=0, \tag{4}$$

whence

$$\frac{\mathfrak{R}_2\times\mathfrak{R}_3}{n_1}=\frac{\mathfrak{R}_1\times\mathfrak{R}_3}{n_2}=\frac{\mathfrak{R}_1\times\mathfrak{R}_2}{n_3}. \tag{5}$$

Our equation may therefore be regarded as signifying that the three vectors $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$ lie in one plane, and that the three triangles determined each by a pair of these vectors, and usually denoted by $[r_2r_3]$, $[r_1r_3]$, $[r_1r_2]$, are proportional to

$$A_1\left(1+\frac{B_1}{r_1^3}\right),\quad \left(1-\frac{B_2}{r_2^3}\right),\quad A_3\left(1+\frac{B_3}{r_3^3}\right).$$

Since this vector equation is equivalent to three ordinary equations, it is evidently sufficient to determine the three positions of the body in connection with the conditions that these positions must lie upon the lines of sight of three observations. To give analytical expression to these conditions, we may write $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$ for the vectors drawn from the sun to the three positions of the earth (or, more exactly, of the observatories where the observations have been made), $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ for unit vectors drawn in the directions of the body, as observed, and ρ_1, ρ_2, ρ_3 for the three distances of the body from the places of observation. We have then

$$\mathfrak{R}_1=\mathfrak{E}_1+\rho_1\mathfrak{F}_1,\quad \mathfrak{R}_2=\mathfrak{E}_2+\rho_2\mathfrak{F}_2,\quad \mathfrak{R}_3=\mathfrak{E}_3+\rho_3\mathfrak{F}_3. \tag{6}$$

By substitution of these values our fundamental equation becomes

$$A_1\left(1+\frac{B_1}{r_1^3}\right)(\mathfrak{E}_1+\rho_1\mathfrak{F}_1)-\left(1-\frac{B_2}{r_2^3}\right)(\mathfrak{E}_2+\rho_2\mathfrak{F}_2)+A_3\left(1+\frac{B_3}{r_3^3}\right)(\mathfrak{E}_3+\rho_3\mathfrak{F}_3)=0, \tag{7}$$

where ρ_1, ρ_2, ρ_3, r_1, r_2, r_3 (the geocentric and heliocentric distances) are the only unknown quantities. From equations (6) we also get, by squaring both members in each,

$$\left.\begin{aligned} r_1^2&=\mathfrak{E}_1^2+2(\mathfrak{E}_1\cdot\mathfrak{F}_1)\rho_1+\rho_1^2,\quad r_2^2=\mathfrak{E}_2^2+2(\mathfrak{E}_2\cdot\mathfrak{F}_2)\rho_2+\rho_2^2,\\ r_3^2&=\mathfrak{E}_3^2+2(\mathfrak{E}_3\cdot\mathfrak{F}_3)\rho_3+\rho_3^2,\end{aligned}\right\} \tag{8}$$

by which the values of r_1, r_2, r_3 may be derived from those of ρ_1, ρ_2, ρ_3, or *vice versâ*. Equations (7) and (8), which are equivalent to six ordinary equations, are sufficient to determine the six quantities r_1, r_2, r_3, ρ_1, ρ_2, ρ_3; or, if we suppose the values of r_1, r_2, r_3 in terms of ρ_1, ρ_2, ρ_3 to be substituted in equation (7), we have a single vector equation, from which we may determine the three geocentric distances ρ_1, ρ_2, ρ_3.

It remains to be shown, first, how the numerical solution of the equation may be performed, and secondly, how such an approximate solution of the actual problem may furnish the basis of a closer approximation.

Solution of the Fundamental Equation.

The relations with which we have to do will be rendered a little more simple if instead of each geocentric distance we introduce the

distance of the body from the foot of the perpendicular from the sun upon the line of sight. If we set

$$q_1=\rho_1+(\mathfrak{E}_1\cdot\mathfrak{F}_1),\qquad q_2=\rho_2+(\mathfrak{E}_2\cdot\mathfrak{F}_2),\qquad q_3=\rho_3+(\mathfrak{E}_3\cdot\mathfrak{F}_3), \tag{9}$$

$$p_1^2=\mathfrak{E}_1^2-(\mathfrak{E}_1\cdot\mathfrak{F}_1)^2,\quad p_2^2=\mathfrak{E}_2^2-(\mathfrak{E}_2\cdot\mathfrak{F}_2)^2,\quad p_3=\mathfrak{E}_3^2-(\mathfrak{E}_3\cdot\mathfrak{F}_3)^2, \tag{10}$$

equations (8) become

$$r_1^2=q_1^2+p_1^2,\qquad r_2^2=q_2^2+p_2^2,\qquad r_3^2=q_3^2+p_3^2. \tag{11}$$

Let us also set, for brevity,

$$\left.\begin{aligned}\mathfrak{S}_1=A_1\Big(1+\frac{B_1}{r_1^3}\Big)(\mathfrak{E}_1+\rho_1\mathfrak{F}_1),\quad \mathfrak{S}_2=-\Big(1-\frac{B_2}{r_2^3}\Big)(\mathfrak{E}_2+\rho_2\mathfrak{F}_2),\\ \mathfrak{S}_3=A_3\Big(1+\frac{B_3}{r_3^3}\Big)(\mathfrak{E}_3+\rho_3\mathfrak{F}_3).\end{aligned}\right\} \tag{12}$$

Then $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ may be regarded as functions respectively of ρ_1, ρ_2, ρ_3, therefore of q_1, q_2, q_3, and if we set

$$\mathfrak{S}'=\frac{d\mathfrak{S}_1}{dq_1},\quad \mathfrak{S}''=\frac{d\mathfrak{S}_2}{dq_2},\quad \mathfrak{S}'''=\frac{d\mathfrak{S}_3}{dq_3}, \tag{13}$$

and

$$\mathfrak{S}=\mathfrak{S}_1+\mathfrak{S}_2+\mathfrak{S}_3, \tag{14}$$

we shall have

$$d\mathfrak{S}=\mathfrak{S}'dq_1+\mathfrak{S}''dq_2+\mathfrak{S}'''dq_3. \tag{15}$$

To determine the value of $\mathfrak{S}'$, we get by differentiation

$$\mathfrak{S}'=A_1\Big(1+\frac{B_1}{r_1^3}\Big)\mathfrak{F}_1-A_1\frac{3B_1}{r_1^4}\frac{dr_1}{dq_1}(\mathfrak{E}_1+\rho_1\mathfrak{F}_1). \tag{16}$$

But by (11)

$$\frac{dr_1}{dq_1}=\frac{q_1}{r_1}. \tag{17}$$

Therefore

$$\left.\begin{aligned}\mathfrak{S}'&=A_1\Big(1+\frac{B_1}{r_1^3}\Big)\mathfrak{F}_1-\frac{3B_1q_1}{r_1^5(1+B_1r_1^{-3})}\mathfrak{S}_1\\ \mathfrak{S}''&=-\Big(1-\frac{B_2}{r_2^3}\Big)\mathfrak{F}_2+\frac{3B_2q_2}{r_2^5(1-B_2r_2^{-3})}\mathfrak{S}_2\\ \mathfrak{S}'''&=A_3\Big(1+\frac{B_3}{r_3^3}\Big)\mathfrak{F}_3-\frac{3B_3q_3}{r_3^5(1+B_3r_3^{-3})}\mathfrak{S}_3\end{aligned}\right\} \tag{18}$$

Now if any values of q_1, q_2, q_3 (either assumed or obtained by a previous approximation) give a certain residual $\mathfrak{S}$ (which would be zero if the values of q_1, q_2, q_3 satisfied the fundamental equation), and we wish to find the corrections Δq_1, Δq_2, Δq_3, which must be added to q_1, q_2, q_3 to reduce the residual to zero, we may apply equation (15) to these finite differences, and will have approximately, when these differences are not very large,

$$-\mathfrak{S}=\mathfrak{S}'\Delta q_1+\mathfrak{S}''\Delta q_2+\mathfrak{S}'''\Delta q_3. \tag{19}$$

This gives *

$$\Delta q_1 = -\frac{(\mathfrak{S}\mathfrak{S}''\mathfrak{S}''')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')} \quad \Delta q_2 = -\frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')} \quad \Delta q_3 = -\frac{(\mathfrak{S}\mathfrak{S}'\mathfrak{S}'')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}. \quad (20)$$

From the corrected values of q_1, q_2, q_3 we may calculate a new residual $\mathfrak{S}$, and from that determine another correction for each of the quantities q_1, q_2, q_3.

It will sometimes be worth while to use formulæ a little less simple for the sake of a more rapid approximation. Instead of equation (19) we may write, with a higher degree of accuracy,

$$-\mathfrak{S} = \mathfrak{S}'\Delta q_1 + \mathfrak{S}''\Delta q_2 + \mathfrak{S}'''\Delta q_3 + \tfrac{1}{2}\mathfrak{T}'(\Delta q_1)^2 + \tfrac{1}{2}\mathfrak{T}''(\Delta q_2)^2 + \tfrac{1}{2}\mathfrak{T}'''(\Delta q_3)^2, \quad (21)$$

where

$$\left.\begin{aligned}
\mathfrak{T}' &= \frac{d^2\mathfrak{S}_1}{dq_1^{\,2}} = 2A_1B_1\frac{d(r_1^{-3})}{dq_1}\mathfrak{F}_1 + \frac{B_1}{1+B_1r_1^{-3}}\frac{d^2(r_1^{-3})}{dq_1^{\,2}}\mathfrak{S}_1 \\
\mathfrak{T}'' &= \frac{d^2\mathfrak{S}_2}{dq_2^{\,2}} = \quad 2B_2\frac{d(r_2^{-3})}{dq_2}\mathfrak{F}_2 - \frac{B_2}{1-B_2r_2^{-3}}\frac{d^2(r_2^{-3})}{dq_2^{\,2}}\mathfrak{S}_2 \\
\mathfrak{T}''' &= \frac{d^2\mathfrak{S}_3}{dq_3^{\,2}} = 2A_3B_3\frac{d(r_3^{-3})}{dq_3}\mathfrak{F}_3 + \frac{B_3}{1+B_3r_3^{-3}}\frac{d^2(r_3^{-3})}{dq_3^{\,2}}\mathfrak{S}_3
\end{aligned}\right\} \quad (22)$$

It is evident that $\mathfrak{T}''$ is generally many times greater than $\mathfrak{T}'$ or $\mathfrak{T}'''$, the factor B_2, in the case of equal intervals, being exactly ten times as great as A_1B_1 or A_3B_3. This shows, in the first place, that the accurate determination of Δq_2 is of the most importance for the subsequent approximations. It also shows that we may attain nearly the same accuracy in writing

$$-\mathfrak{S} = \mathfrak{S}'\Delta q_1 + \mathfrak{S}''\Delta q_2 + \mathfrak{S}'''\Delta q_3 + \tfrac{1}{2}\mathfrak{T}''\Delta q_2^{\,2}. \quad (23)$$

We may, however, often do a little better than this without using a more complicated equation. For $\mathfrak{T}' + \mathfrak{T}'''$ may be estimated very roughly as equal to $\frac{1}{5}\mathfrak{T}''$. Whenever, therefore, Δq_1 and Δq_3 are about as large as Δq_2, as is often the case, it may be a little better to use the coefficient $\frac{6}{10}$ instead of $\frac{1}{2}$ in the last term.

For Δq_2, then, we have the equation

$$-(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}') = (\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')\Delta q_2 + \tfrac{6}{10}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')\Delta q_2^{\,2}. \quad (24)$$

$(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')$ is easily computed from the formula

$$\left.\begin{aligned}
(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}') = \frac{1}{q_2}\left(1-5\frac{q_2^{\,2}}{r_2^{\,2}}\right)\left((\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''') + (\mathfrak{F}_2\mathfrak{S}'''\mathfrak{S}')\right) \\
-\frac{B_2}{q_2r_2^{\,3}}\left(1+\frac{q_2^{\,2}}{r_2^{\,2}}\right)(\mathfrak{F}_2\mathfrak{S}'''\mathfrak{S}'),
\end{aligned}\right\} \quad (25)$$

which may be derived from equations (18) and (22).

* These equations are obtained by taking the direct products of both members of the preceding equation with $\mathfrak{S}'' \times \mathfrak{S}'''$, $\mathfrak{S}''' \times \mathfrak{S}'$, and $\mathfrak{S}' \times \mathfrak{S}''$, respectively. See footnote on page 119.

The quadratic equation (24) gives two values of the correction to be applied to the position of the body. When they are not too large, they will belong to two different solutions of the problem, generally to the two least removed from the values assumed. But a very large value of Δq_2 must not be regarded as affording any trustworthy indication of a solution of the problem. In the majority of cases we only care for one of the roots of the equation, which is distinguished by being very small, and which will be most easily calculated by a small correction to the value which we get by neglecting the quadratic term.*

When a comet is somewhat near the earth we may make use of the fact that the earth's orbit is one solution of the problem, i.e., that $-\rho_2$ is one value of Δq_2, to save the trifling labor of computing the value of $(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')$. For it is evident from the theory of equations that if $-\rho_2$ and z are the two roots,

$$\rho_2 - z = \frac{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}{\frac{3}{5}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')} \qquad -\rho_2 z = \frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{\frac{3}{5}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')}.$$

Eliminating $(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')$, we have

$$(\rho_2 - z)(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}') = -\rho_2 z(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}'''),$$

whence

$$\frac{1}{z} = \frac{1}{\rho_2} - \frac{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}.$$

Now $-\dfrac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}$ is the value of Δq_2, which we obtain if we neglect the quadratic term in equation (24). If we call this value $[\Delta q_2]$, we have for the more exact value †

$$\Delta q_2 = \frac{[\Delta q_2]}{1 + \dfrac{[\Delta q_2]}{\rho_2}}. \tag{26}$$

The quantities Δq_1 and Δq_3 might be calculated by the equations

$$\left.\begin{aligned} -(\mathfrak{S}\mathfrak{S}''\mathfrak{S}''') &= (\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')\Delta q_1 + \tfrac{6}{10}(\mathfrak{T}''\mathfrak{S}''\mathfrak{S}''')\Delta q_2{}^2 \\ -(\mathfrak{S}\mathfrak{S}'\mathfrak{S}'') &= (\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')\Delta q_3 + \tfrac{6}{10}(\mathfrak{T}''\mathfrak{S}'\mathfrak{S}'')\Delta q_2{}^2 \end{aligned}\right\} \tag{27}$$

* In the case of Swift's comet (V, 1880), the writer found by the quadratic equation $-\cdot247$ and $-\cdot116$ for corrections of the assumed geocentric distance $\cdot250$. The first of these numbers gives an approximation to the position of the earth; the second to that of the comet, viz., the geocentric distance $\cdot134$ instead of the true value $\cdot1333$. The coefficient $\frac{6}{10}$ was used in the quadratic equation; with the coefficient $\frac{1}{2}$ the approximations would not be quite so good. The value of the correction obtained by neglecting the quadratic term was $\cdot079$, which indicates that the approximations (in this very critical case) would be quite tedious without the use of the quadratic term.

† In the case mentioned in the preceding footnote, from $[\Delta q_2] = -\cdot079$ and $\rho_2 = \cdot25$, we get $\Delta q_2 = -\cdot1155$, which is sensibly the same value as that obtained by calculating the quadratic term.

But a little examination will show that the coefficients of Δq_2^2 in these equations will not generally have very different values from the coefficient of the same quantity in equation (24). We may therefore write with sufficient accuracy

$$\Delta q_1 = [\Delta q_1] + \Delta q_2 - [\Delta q_2], \quad \Delta q_3 = [\Delta q_3] + \Delta q_2 - [\Delta q_2], \qquad (28)$$

where $[\Delta q_1], [\Delta q_2], [\Delta q_3]$ denote the values obtained from equations (20).

In making successive corrections of the distances q_1, q_2, q_3, it will not be necessary to recalculate the values of $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$, when these have been calculated from fairly good values of q_1, q_2, q_3. But when, as is generally the case, the first assumption is only a rude guess, the values of $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$ should be recalculated after one or two corrections of q_1, q_2, q_3. To get the best results when we do not recalculate $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$, we may proceed as follows: Let $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$ denote the values which have been calculated; Dq_1, Dq_2, Dq_3, respectively, the sum of the corrections of each of the quantities q_1, q_2, q_3, which have been made since the calculation of $\mathfrak{S}'$, $\mathfrak{S}''$, $\mathfrak{S}'''$; $\mathfrak{S}$ the residual after all the corrections of q_1, q_2, q_3, which have been made; and Δq_1, Δq_2, Δq_3 the remaining corrections which we are seeking. We have, then, very nearly

$$\left.\begin{aligned} -\mathfrak{S} = \{\mathfrak{S}' + \mathfrak{T}'(Dq_1 + \tfrac{1}{2}\Delta q_1)\}\Delta q_1 + \{\mathfrak{S}'' + \mathfrak{T}''(Dq_2 + \tfrac{1}{2}\Delta q_2)\}\Delta q_2 \\ + \{\mathfrak{S}''' + \mathfrak{T}'''(Dq_3 + \tfrac{1}{2}\Delta q_3)\}\Delta q_3. \end{aligned}\right\} \qquad (29)$$

The same considerations which we applied to equation (21) enable us to simplify this equation also, and to write with a fair degree of accuracy

$$-(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}') = \{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''') + \tfrac{4}{5}(\mathfrak{T}''\mathfrak{S}'''\mathfrak{S}')(Dq_2 + \tfrac{1}{2}\Delta q_2)\}\Delta q_2, \qquad (30)$$

$$\Delta q_1 = [\Delta q_1] + \Delta q_2 - [\Delta q_2], \quad \Delta q_3 = [\Delta q_3] + \Delta q_2 - [\Delta q_2], \qquad (31)$$

where

$$[\Delta q_1] = -\frac{(\mathfrak{S}\mathfrak{S}''\mathfrak{S}''')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}, \quad [\Delta q_2] = -\frac{(\mathfrak{S}\mathfrak{S}'''\mathfrak{S}')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}, \quad [\Delta q_3] = -\frac{(\mathfrak{S}\mathfrak{S}'\mathfrak{S}'')}{(\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''')}. \qquad (32)$$

Correction of the Fundamental Equation.

When we have thus determined, by the numerical solution of our fundamental equation, approximate values of the three positions of the body, it will always be possible to apply a small numerical correction to the equation, so as to make it agree exactly with the laws of elliptic motion in a fictitious case differing but little from the actual. After such a correction the equation will evidently apply to the actual case with a much higher degree of approximation.

There is room for great diversity in the application of this principle. The method which appears to the writer the most simple and direct is the following, in which the correction of the intervals for aberration

is combined with the correction required by the approximate nature of the equation.*

The solution of the fundamental equation gives us three points, which must necessarily lie in one plane with the sun, and in the lines of sight of the several observations. Through these points we may pass an ellipse, and calculate the intervals of time required by the exact laws of elliptic motion for the passage of the body between them. If these calculated intervals should be identical with the given intervals, corrected for aberration, we would evidently have the true solution of the problem. But suppose, to fix our ideas, that the calculated intervals are a little too long. It is evident that if we repeat our calculations, using in our fundamental equation intervals shortened in the same ratio as the calculated intervals have come out too long, the intervals calculated from the second solution of the fundamental equation must agree almost exactly with the desired values. If necessary, this process may be repeated, and thus any required degree of accuracy may be obtained, whenever the solution of the uncorrected equation gives an approximation to the true positions. For this it is necessary that the intervals should not be too great. It appears, however, from the results of the example of Ceres, given hereafter, in which the heliocentric motion exceeds 62° but the calculated values of the intervals of time differ from the given values by little more than one part in two thousand, that we have here not approached the limit of the application of our formula.

In the usual terminology of the subject, the fundamental equation with intervals uncorrected for aberration represents the *first hypothesis;* the same equation with the intervals affected by certain numerical coefficients (differing little from unity) represents the *second hypothesis;* the *third hypothesis*, should such be necessary, is represented by a similar equation with corrected coefficients, etc.

In the process indicated there are certain economies of labor which should not be left unmentioned, and certain precautions to be observed in order that the neglected figures in our computations may not unduly influence the result.

It is evident, in the first place, that for the correction of our fundamental equation we need not trouble ourselves with the position of the orbit in the solar system. The intervals of time, which determine this correction, depend only on the three heliocentric distances r_1, r_2, r_3 and the two heliocentric angles, which will be represented by $v_2 - v_1$ and $v_3 - v_2$, if we write v_1, v_2, v_3 for the true anomalies. These angles ($v_2 - v_1$ and $v_3 - v_2$) may be determined from r_1, r_2, r_3 and n_1, n_2, n_3,

* When an approximate orbit is known in advance, we may correct the fundamental equation at once. The formulæ will be given in the Summary, § xii.

and therefore from r_1, r_2, r_3 and the given intervals. For our fundamental equation, which may be written

$$n_1\Re_1 - n_2\Re_2 + n_3\Re_3 = 0, \tag{33}$$

indicates that we may form a triangle in which the lengths of the sides shall be n_1r_1, n_2r_2, and n_3r_3 (let us say for brevity, s_1, s_2, s_3), and the directions of the sides parallel with the three heliocentric directions of the body. The angles opposite s_1 and s_3 will be respectively $v_3 - v_2$ and $v_2 - v_1$. We have, therefore, by a well-known formula,

$$\left.\begin{aligned} \tan\frac{v_3-v_2}{2} &= \sqrt{\frac{(s_1-s_2+s_3)(s_1+s_2-s_3)}{(s_1+s_2+s_3)(-s_1+s_2+s_3)}} \\ \tan\frac{v_2-v_1}{2} &= \sqrt{\frac{(-s_1+s_2+s_3)(s_1-s_2-s_3)}{(s_1+s_2+s_3)(s_1+s_2-s_3)}} \end{aligned}\right\}. \tag{34}$$

As soon, therefore, as the solution of our fundamental equation has given a sufficient approximation to the values of r_1, r_2, r_3 (say five- or six-figure values, if our final result is to be as exact as seven-figure logarithms can make it), we calculate n_1, n_2, n_3 with seven-figure logarithms by equations (2), and the heliocentric angles by equations (34).

The semi-parameter corresponding to these values of the heliocentric distances and angles is given by the equation

$$p = \frac{n_1r_1 - n_2r_2 + n_3r_3}{n_1 - n_2 + n_3}. \tag{35}$$

The expression $n_1 - n_2 + n_3$, which occurs in the value of the semi-parameter, and the expression $n_1r_1 - n_2r_2 + n_3r_3$, or $s_1 - s_2 + s_3$, which occurs both in the value of the semi-parameter and in the formulæ for determining the heliocentric angles, represent small quantities of the second order (if we call the heliocentric angles small quantities of the first order), and cannot be very accurately determined from approximate numerical values of their separate terms. The first of these quantities may, however, be determined accurately by the formula

$$n_1 - n_2 + n_3 = \frac{A_1B_1}{r_1^3} + \frac{B_2}{r_2^3} + \frac{A_3B_3}{r_3^3}. \tag{36}$$

With respect to the quantity $s_1 - s_2 + s_3$, a little consideration will show that if we are careful to use the same value wherever the expression occurs, both in the formulæ for the heliocentric angles and for the semi-parameter, the inaccuracy of the determination of this value from the cause mentioned will be of no consequence in the process of correcting the fundamental equation. For although the logarithm of $s_1 - s_2 + s_3$ as calculated by seven-figure logarithms from r_1, r_2, r_3 may be accurate only to four or five figures, we may regard it as absolutely correct if we make a very small change in the value of one

of the heliocentric distances (say r_2). We need not trouble ourselves farther about this change, for it will be of a magnitude which we neglect in computations with seven-figure tables. That the heliocentric angles thus determined may not agree as closely as they might with the positions on the lines of sight determined by the first solution of the fundamental equation is of no especial consequence in the correction of the fundamental equation, which only requires the exact fulfilment of two conditions, viz., that our values of the heliocentric distances and angles shall have the relations required by the fundamental equation to the given intervals of time, and that they shall have the relations required by the exact laws of elliptic motion to the calculated intervals of time. The third condition, that none of these values shall differ too widely from the actual values, is of a looser character.

After the determination of the heliocentric angles and the semi-parameter, the eccentricity and the true anomalies of the three positions may next be determined, and from these the intervals of time. These processes require no especial notice. The appropriate formulæ will be given in the Summary of Formulæ.

Determination of the Orbit from the Three Positions and the Intervals of Time.

The values of the semi-parameter and the heliocentric angles as given in the preceding paragraphs depend upon the quantity $s_1 - s_2 + s_3$, the numerical determination of which from s_1, s_2, and s_3 is critical to the second degree when the heliocentric angles are small. This was of no consequence in the process which we have called the *correction of the fundamental equation*. But for the actual determination of the orbit from the positions given by the corrected equation—or by the uncorrected equation, when we judge that to be sufficient—a more accurate determination of this quantity will generally be necessary. This may be obtained in different ways, of which the following is perhaps the most simple. Let us set

$$\mathfrak{S}_4 = \mathfrak{S}_3 - \mathfrak{S}_1, \tag{37}$$

and s_4 for the length of the vector $\mathfrak{S}_4$, obtained by taking the square root of the sum of the squares of the components of the vector. It is evident that s_2 is the longer and s_4 the shorter diagonal of a parallelogram of which the sides are s_1 and s_3. The area of the triangle having the sides s_1, s_2, s_3 is therefore equal to that of the triangle having the sides s_1, s_3, s_4, each being one-half of the parallelogram. This gives

$$\begin{aligned}(s_1+s_2+s_3)(-s_1+s_2+s_3)(s_1-s_2+s_3)(s_1+s_2-s_3)\\ =(s_1+s_4+s_3)(-s_1+s_4+s_3)(s_1-s_4+s_3)(s_1+s_4-s_3),\end{aligned} \tag{38}$$

and

$$s_1-s_2+s_3=\frac{(s_1+s_4+s_3)(-s_1+s_4+s_3)(s_1-s_4+s_3)(s_1+s_4-s_3)}{(s_1+s_2+s_3)(-s_1+s_2+s_3)(s_1+s_2-s_3)}. \quad (39)$$

The numerical determination of this value of $s_1-s_2+s_3$ is critical only to the first degree.

The eccentricity and the true anomalies may be determined in the same way as in the correction of the formula. The position of the orbit in space may be derived from the following considerations. The vector $-\mathfrak{S}_2$ is directed from the sun toward the second position of the body; the vector $\mathfrak{S}_4$ from the first to the third position. If we set

$$\mathfrak{S}_5=\mathfrak{S}_4-\frac{\mathfrak{S}_4\cdot\mathfrak{S}_2}{s_2^2}\mathfrak{S}_2, \quad (40)$$

the vector $\mathfrak{S}_5$ will be in the plane of the orbit, perpendicular to $-\mathfrak{S}_2$ and on the side toward which anomalies increase. If we write s_5 for the length of $\mathfrak{S}_5$,

$$-\frac{\mathfrak{S}_2}{s_2} \text{ and } \frac{\mathfrak{S}_5}{s_5}$$

will be unit vectors. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be unit vectors determining the position of the orbit, $\mathfrak{P}$ being drawn from the sun toward the perihelion, and $\mathfrak{P}'$ at right angles to $\mathfrak{P}$, in the plane of the orbit, and on the side toward which anomalies increase. Then

$$\mathfrak{P}=-\cos v_2\frac{\mathfrak{S}_2}{s_2}-\sin v_2\frac{\mathfrak{S}_5}{s_5}, \quad (41)$$

$$\mathfrak{P}'=-\sin v_2\frac{\mathfrak{S}_2}{s_2}+\cos v_2\frac{\mathfrak{S}_5}{s_5}. \quad (42)$$

The time of perihelion passage (T) may be determined from any one of the observations by the equation

$$\frac{k}{a^{\frac{3}{2}}}(t-T)=E-e\sin E, \quad (43)$$

the eccentric anomaly E being calculated from the true anomaly v. The interval $t-T$ in this equation is to be measured in days. A better value of T may be found by averaging the three values given by the separate observations, with such weights as the circumstances may suggest. But any considerable differences in the three values of T would indicate the necessity of a second correction of the formula, and furnish the basis for it.

For the calculation of an ephemeris we have

$$\mathfrak{R}=-ae\mathfrak{P}+\cos E\,a\mathfrak{P}+\sin E\,b\mathfrak{P}' \quad (44)$$

in connection with the preceding equation.

Sometimes it may be worth while to make the calculations for the correction of the formula in the slightly longer form indicated for

the determination of the orbit. This will be the case when we wish simultaneously to correct the formula for its theoretical imperfection, and to correct the observations by comparison with others not too remote. The rough approximation to the orbit given by the uncorrected formula may be sufficient for this purpose. In fact, for observations separated by very small intervals, the imperfection of the uncorrected formula will be likely to affect the orbit less than the errors of the observations.

The computer may prefer to determine the orbit from the first and third heliocentric positions with their times. This process, which has certain advantages, is perhaps a little longer than that here given, and does not lend itself quite so readily to successive improvements of the hypothesis. When it is desired to derive an improved hypothesis from an orbit thus determined, the formulæ in § XII of the summary may be used.

SUMMARY OF FORMULÆ

WITH DIRECTIONS FOR USE.

(For the case in which an approximate orbit is known in advance, see XII.)

I.

Preliminary computations relating to the intervals of time.

t_1, t_2, t_3 = times of the observations in days.

$\log k = 8{\cdot}2355814$ (after Gauss)

$$\tau_1 = k(t_3 - t_2) \qquad \tau_3 = k(t_2 - t_1)$$

$$A_1 = \frac{t_3 - t_2}{t_3 - t_1} \qquad A_3 = \frac{t_2 - t_1}{t_3 - t_1}$$

$$B_1 = \frac{-\tau_1^2 + \tau_1\tau_3 + \tau_3^2}{12} \qquad B_2 = \frac{\tau_1^2 + 3\tau_1\tau_3 + \tau_3^2}{12} \qquad B_3 = \frac{\tau_1^2 + \tau_1\tau_3 - \tau_3^2}{12}$$

For control: $A_1B_1 + B_2 + A_3B_3 = \frac{1}{2}\tau_1\tau_3$.

II.

Preliminary computations relating to the first observation.

X_1, Y_1, Z_1 (components of $\mathfrak{E}_1$) = the heliocentric coordinates of the earth, increased by the geocentric coordinates of the observatory.

ξ_1, η_1, ζ_1 (components of $\mathfrak{F}_1$) = the direction-cosines of the observed position, corrected for the aberration of the fixed stars.

$$\mathfrak{E}_1^2 = X_1^2 + Y_1^2 + Z_1^2 \qquad (\mathfrak{E}_1.\mathfrak{F}_1) = X_1\xi_1 + Y_1\eta_1 + Z_1\zeta_1 \qquad p_1^2 = \mathfrak{E}_1^2 - (\mathfrak{E}_1.\mathfrak{F}_1)^2$$

Preliminary computations relating to the second and third observations.

The formulæ are entirely analogous to those relating to the first observation, the quantities being distinguished by the proper suffixes.

III.

Equations of the first hypothesis.

When the preceding quantities have been computed, their numerical values (or their logarithms, when more convenient for computation,) are to be substituted in the following equations:

Components of $\mathfrak{S}_1$

$$\left.\begin{aligned} q_1 &= \rho_1 + (\mathfrak{E}_1 \cdot \mathfrak{F}_1) & \alpha_1 &= A_1\xi_1(1+R_1)\left(q_1 + \frac{X_1}{\xi_1} - (\mathfrak{E}_1 \cdot \mathfrak{F}_1)\right) \\ r_1^2 &= q_1^2 + p_1^2 & \beta_1 &= A_1\eta_1(1+R_1)\left(q_1 + \frac{Y_1}{\eta_1} - (\mathfrak{E}_1 \cdot \mathfrak{F}_1)\right) \\ R_1 &= \frac{B_1}{r_1^3} & \gamma_1 &= A_1\zeta_1(1+R_1)\left(q_1 + \frac{Z_1}{\zeta_1} - (\mathfrak{E}_1 \cdot \mathfrak{F}_1)\right) \end{aligned}\right\} \text{III}_1$$

For control: $s_1^2 = \alpha_1^2 + \beta_1^2 + \gamma_1^2 = A_1^2(1+R_1)^2 r_1^2$

Components of $\mathfrak{S}'$

$$P' = \frac{3R_1q_1}{(1+R_1)r_1^2} \qquad \left.\begin{aligned} \alpha' &= A_1\xi_1 + A_1\xi_1R_1 - P'a_1 \\ \beta' &= A_1\eta_1 + A_1\eta_1R_1 - P'\beta_1 \\ \gamma' &= A_1\zeta_1 + A_1\zeta_1R_1 - P'\gamma_1 \end{aligned}\right\} \text{III}'$$

Components of $\mathfrak{S}_2$

$$\left.\begin{aligned} q_2 &= \rho_2 + (\mathfrak{E}_2 \cdot \mathfrak{F}_2) & \alpha_2 &= -\xi_2(1-R_2)\left(q_2 + \frac{X_2}{\xi_2} - (\mathfrak{E}_2 \cdot \mathfrak{F}_2)\right) \\ r_2^2 &= q_2^2 + p_2^2 & \beta_2 &= -\eta_2(1-R_2)\left(q_2 + \frac{Y_2}{\eta_2} - (\mathfrak{E}_2 \cdot \mathfrak{F}_2)\right) \\ R_2 &= \frac{B_2}{r_2^3} & \gamma_2 &= -\zeta_2(1-R_2)\left(q_2 + \frac{Z_2}{\zeta_2} - (\mathfrak{E}_2 \cdot \mathfrak{F}_2)\right) \end{aligned}\right\} \text{III}_2$$

For control: $s_2^2 = \alpha_2^2 + \beta_2^2 + \gamma_2^2 = (1-R_2)^2 r_2^2$

Components of $\mathfrak{S}''$

$$P'' = \frac{3R_2q_2}{(1-R_2)r_2^2} \qquad \left.\begin{aligned} \alpha'' &= -\xi_2 + \xi_2R_2 + P''a_2 \\ \beta'' &= -\eta_2 + \eta_2R_2 + P''\beta_2 \\ \gamma'' &= -\zeta_2 + \zeta_2R_2 + P''\gamma_2 \end{aligned}\right\} \text{III}''$$

Components of $\mathfrak{S}_3$

$$\left.\begin{aligned} q_3 &= \rho_3 + (\mathfrak{E}_3 \cdot \mathfrak{F}_3) & \alpha_3 &= A_3\xi_3(1+R_3)\left(q_3 + \frac{X_3}{\xi_3} - (\mathfrak{E}_3 \cdot \mathfrak{F}_3)\right) \\ r_3^2 &= q_3^2 + p_3^2 & \beta_3 &= A_3\eta_3(1+R_3)\left(q_3 + \frac{Y_3}{\eta_3} - (\mathfrak{E}_3 \cdot \mathfrak{F}_3)\right) \\ R_3 &= \frac{B_3}{r_3^3} & \gamma_3 &= A_3\zeta_3(1+R_3)\left(q_3 + \frac{Z_3}{\zeta_3} - (\mathfrak{E}_3 \cdot \mathfrak{F}_3)\right) \end{aligned}\right\} \text{III}_3$$

For control: $$\varepsilon_3^2 = a_3^2 + \beta_3^2 + \gamma_3^2 = A_3^2(1+R_3)^2 r_3^2$$

$$P''' = \frac{3R_3 q_3}{(1+R_3)r_3^2}$$

Components of $\mathfrak{S}'''$

$$\left.\begin{aligned} a''' &= A_3\xi_3 + A_3\xi_3 R_3 - P'''a_3 \\ \beta''' &= A_3\eta_3 + A_3\eta_3 R_3 - P'''\beta_3 \\ \gamma''' &= A_3\zeta_3 + A_3\zeta_3 R_3 - P'''\gamma_3 \end{aligned}\right\} \text{III}'''$$

The computer is now to assume any reasonable values either of the geocentric distances, ρ_1, ρ_2, ρ_3, or of the heliocentric distances, r_1, r_2, r_3 (the former in the case of a comet, the latter in the case of an asteroid), and from these assumed values to compute the rest of the following quantities:

By equations III_1, III′.	By equations III_2, III″.	By equations III_3, III‴.
q_1	q_2	q_3
$\log r_1$	$\log r_2$	$\log r_3$
$\log R_1$	$\log R_2$	$\log R_3$
$\log(1+R_1)$	$\log(1-R_2)$	$\log(1+R_3)$
$\log P'$	$\log P''$	$\log P'''$
a_1	a_2	a_3
β_1	β_2	β_3
γ_1	γ_2	γ_3
a'	a''	a'''
β'	β''	β'''
γ'	γ''	γ'''

IV.

Calculations relating to differential coefficients.

Components of $\mathfrak{S}''\times\mathfrak{S}'''$	Components of $\mathfrak{S}'''\times\mathfrak{S}'$	Components of $\mathfrak{S}'\times\mathfrak{S}''$
$a_1 = \beta''\gamma''' - \gamma''\beta'''$	$a_2 = \beta'''\gamma' - \gamma'''\beta'$	$a_3 = \beta'\gamma'' - \gamma'\beta''$
$b_1 = \gamma''a''' - a''\gamma'''$	$b_2 = \gamma'''a' - a'''\gamma'$	$b_3 = \gamma'a'' - a'\gamma''$
$c_1 = a''\beta''' - \beta''a'''$	$c_2 = a'''\beta' - \beta'''a'$	$c_3 = a'\beta'' - \beta'a''$

$$G = (\mathfrak{S}'\mathfrak{S}''\mathfrak{S}''') = a_1a' + b_1\beta' + c_1\gamma' = a_2a'' + b_2\beta'' + c_2\gamma'' = a_3a''' + b_3\beta''' + c_3\gamma'''.$$

These computations are controlled by the agreement of the three values of G.

The following are not necessary except when the corrections to be made are large:

$$H = (\mathfrak{d}_2\mathfrak{S}'''\mathfrak{S}') = a_2\xi_2 + b_2\eta_2 + c_2\zeta_2$$

$$L = \frac{1}{q_2}\left(1+\frac{H}{G}\right)\left(1-5\frac{q_2^2}{r_2^2}\right) - \frac{R_2 H}{q_2 G}\left(1+\frac{q_2^2}{r_2^2}\right)$$

V.

Corrections of the geocentric distances.

Components of $\mathfrak{S}$.

$$\alpha=\alpha_1+\alpha_2+\alpha_3 \qquad C_1=-\frac{a_1\alpha+b_1\beta+c_1\gamma}{G}$$

$$\beta=\beta_1+\beta_2+\beta_3 \qquad C_2=-\frac{a_2\alpha+b_2\beta+c_2\gamma}{G}$$

$$\gamma=\gamma_1+\gamma_2+\gamma_3 \qquad C_3=-\frac{a_3\alpha+b_3\beta+c_3\gamma}{G}$$

$$\Delta q_2=C_2-\tfrac{6}{10}L(\Delta q_2)^2.$$

(This equation will generally be most easily solved by repeated substitutions.)

$$\Delta q_1=C_1-\tfrac{6}{10}L(\Delta q_2)^2 \qquad \Delta q_3=C_3-\tfrac{6}{10}L(\Delta q_2)^2.$$

VI.

Successive corrections.

Δq_1, Δq_2, Δq_3 are to be added as corrections to q_1, q_2, q_3. With the new values thus obtained the computation by equations III_1, III_2, III_3 are to be recommenced. Two courses are now open:

(*a*) The work may be carried on exactly as before to the determination of new corrections for q_1, q_2, q_3.

(*b*) The computations by equations III′, III″, III‴, and IV may be omitted, and the old values of a_1, b_1, c_1, a_2, etc., G, and L may be used with the new residuals α, β, γ to get new corrections for q_1, q_2, q_3 by the equations

$$\Delta q_2=\frac{C_2}{1+\frac{6}{5}L(Dq_2+\frac{1}{2}C_2)},$$

$$\Delta q_1=C_1+\Delta q_2-C_2, \qquad \Delta q_3=C_3+\Delta q_2-C_2,$$

where Dq_2 denotes the former correction of q_2. (More generally, at any stage of the work, Dq_2 will represent the sum of all the corrections of q_2 which have been made since the last computation of a_1, b_1, etc.)

So far as any general rule can be given, it is advised to recompute a_1, b_1, etc., and G *once*, perhaps after the second corrections of q_1, q_2, q_3, unless the assumed values represent a fair approximation. Whether L is also to be recomputed, depends on its magnitude, and on that of the correction of q_2, which remains to be made. In the later stages of the work, when the corrections are small, the terms containing L may be neglected altogether.

The corrections of q_1, q_2, q_3 should be repeated until the equations

$$\alpha=0 \qquad \beta=0 \qquad \gamma=0$$

are nearly satisfied. Approximate values of r_1, r_2, r_3 may suffice for the following computations, which, however, must be made with the greatest exactness.

VII.

Test of the first hypothesis.

$\log r_1$, $\log r_2$, $\log r_3$ (approximate values from the preceding computations).

$$N = A_1B_1r_1^{-3} + B_2r_2^{-3} + A_3B_3r_3^{-3}$$
$$s_1 = A_1r_1 + A_1B_1r_1^{-2}$$
$$s_2 = r_2 - B_2r_2^{-2}$$
$$s_3 = A_3r_3 + A_3B_3r_3^{-2}$$
$$s = \tfrac{1}{2}(s_1 + s_2 + s_3)$$
$$s - s_1,\ s - s_2,\ s - s_3.$$

The value of $s - s_2$ may be very small, and its logarithm in consequence ill determined. This will do no harm if the computer is careful to use the same value—computed, of course, as carefully as possible—wherever the expression occurs in the following formulæ:

$$\mathrm{R} = \sqrt{\frac{(s-s_1)(s-s_2)(s-s_3)}{s}} \qquad \tan\tfrac{1}{2}(v_2 - v_1) = \frac{\mathrm{R}}{s - s_3}$$
$$p = \frac{2(s-s_2)}{N} \qquad \tan\tfrac{1}{2}(v_3 - v_2) = \frac{\mathrm{R}}{s - s_1}$$
$$\tan\tfrac{1}{2}(v_3 - v_1) = \frac{s - s_2}{\mathrm{R}}$$

For adjustment of values: $\tfrac{1}{2}(v_3 - v_1) = \tfrac{1}{2}(v_2 - v_1) + \tfrac{1}{2}(v_3 - v_2)$

$$e \sin\tfrac{1}{2}(v_3 + v_1) = \frac{\frac{p}{r_1} - \frac{p}{r_3}}{2\sin\tfrac{1}{2}(v_3 - v_1)}$$

$$e \cos\tfrac{1}{2}(v_3 + v_1) = \frac{\frac{p}{r_1} + \frac{p}{r_3} - 2}{2\cos\tfrac{1}{2}(v_3 - v_1)}$$

$$\tan\tfrac{1}{2}(v_3 + v_1) \qquad e^2$$

For control:
$$e \cos v_2 = \frac{p}{r_2} - 1$$

$$\epsilon = \sqrt{\frac{1-e}{1+e}} \qquad a = \frac{p}{1-e^2}$$

$$\tan\tfrac{1}{2}E_1 = \epsilon \tan\tfrac{1}{2}v_1 \qquad \tan\tfrac{1}{2}E_2 = \epsilon \tan\tfrac{1}{2}v_2 \qquad \tan\tfrac{1}{2}E_3 = \epsilon \tan\tfrac{1}{2}v_3$$
$$\tau_{1\ \mathrm{calc.}} = a^{\frac{3}{2}}(E_3 - E_2) + ea^{\frac{3}{2}} \sin E_2 - ea^{\frac{3}{2}} \sin E_3$$
$$\tau_{3\ \mathrm{calc.}} = a^{\frac{3}{2}}(E_2 - E_1) + ea^{\frac{3}{2}} \sin E_1 - ea^{\frac{3}{2}} \sin E_2$$

VIII.

For the second hypothesis.

$\delta\tau_1 = \cdot 0057613k(\rho_2 - \rho_3)$ (aberration-constant after Struve.)

$\delta\tau_3 = \cdot 0057613k(\rho_1 - \rho_2)$ $\log(\cdot 0057613k) = 5\cdot 99610$

$\Delta \log \tau_1 = \log \tau_1 - \log(\tau_{1\,\text{calc.}} - \delta\tau_1)$

$\Delta \log \tau_3 = \log \tau_3 - \log(\tau_{3\,\text{calc.}} - \delta\tau_3)$

$$\Delta \log(\tau_1\tau_3) = \Delta \log \tau_1 + \Delta \log \tau_3$$

$$\Delta \log \frac{\tau_1}{\tau_3} = \Delta \log \tau_1 - \Delta \log \tau_3$$

$$\Delta \log A_1 = -A_3 \Delta \log \frac{\tau_1}{\tau_3}$$

$$\Delta \log A_3 = -A_1 \Delta \log \frac{\tau_1}{\tau_3}$$

$$\Delta \log B_1 = \Delta \log(\tau_1\tau_3) - \frac{\tau_1^2 + \tau_3^2}{12B_1} \Delta \log \frac{\tau_1}{\tau_3}$$

$$\Delta \log B_2 = \Delta \log(\tau_1\tau_3) + \frac{\tau_1^2 - \tau_3^2}{12B_2} \Delta \log \frac{\tau_1}{\tau_3}$$

$$\Delta \log B_3 = \Delta \log(\tau_1\tau_3) + \frac{\tau_1^2 + \tau_3^2}{12B_3} \Delta \log \frac{\tau_1}{\tau_3}$$

These corrections are to be added to the logarithms of A_1, A_3, B_1, B_2, B_3, in equations III_1, III_2, III_3, and the corrected equations used to correct the values of q_1, q_2, q_3, until the residuals α, β, γ vanish. The new values of A_1, A_3 must satisfy the relation $A_1 + A_3 = 1$, and the corrections $\Delta \log A_1$, $\Delta \log A_3$ must be adjusted, if necessary, for this end.

Third hypothesis.

A second correction of equations III_1, III_2, III_3 may be obtained in the same manner as the first, but this will rarely be necessary.

IX.

Determination of the ellipse.

It is supposed that the values of

$$\alpha_1,\ \beta_1,\ \gamma_1, \qquad \alpha_2,\ \beta_2,\ \gamma_2, \qquad \alpha_3,\ \beta_3,\ \gamma_3,$$

$$r_1,\ r_2,\ r_3, \qquad R_1, R_2, R_3, \qquad s_1,\ s_2,\ s_3,$$

have been computed by equations III_1, III_2, III_3 with the greatest exactness, so as to make the residuals α, β, γ vanish, and that the two formulæ for each of the quantities s_1, s_2, s_3 give sensibly the same value.

Components of $\mathfrak{S}_4$	Components of $\mathfrak{S}_5$
$\alpha_4=\alpha_3-\alpha_1$	$\alpha_5=\alpha_4-\frac{\alpha_4\alpha_2+\beta_4\beta_2+\gamma_4\gamma_2}{s_2^2}\alpha_2$
$\beta_4=\beta_3-\beta_1$	$\beta_5=\beta_4-\frac{\alpha_4\alpha_2+\beta_4\beta_2+\gamma_4\gamma_2}{s_2^2}\beta_2$
$\gamma_4=\gamma_3-\gamma_1$	$\gamma_5=\gamma_4-\frac{\alpha_4\alpha_2+\beta_4\beta_2+\gamma_4\gamma_2}{s_2^2}\gamma_2$
$s_4^2=\alpha_4^2+\beta_4^2+\gamma_4^2$	$s_5^2=\alpha_5^2+\beta_5^2+\gamma_5^2$
$s=\frac{1}{2}(s_1+s_2+s_3)$	$S=\frac{1}{2}(s_1+s_4+s_3)$

For control only:

$$s-s_2=\frac{S(S-s_1)(S-s_4)(S-s_3)}{s(s-s_1)(s-s_3)}$$

$$\mathrm{R}^2=\frac{S(S-s_1)(S-s_4)(S-s_3)}{s_2^2} \qquad \tan\tfrac{1}{2}(v_2-v_1)=\frac{\mathrm{R}}{s-s_3}$$

$$N=A_1R_1+R_2+A_3R_3 \qquad \tan\tfrac{1}{2}(v_3-v_2)=\frac{\mathrm{R}}{s-s_1}$$

$$p=\frac{2\mathrm{R}^2 s}{N(s-s_1)(s-s_3)} \qquad \tan\tfrac{1}{2}(v_3-v_1)=\frac{\mathrm{R}s}{(s-s_1)(s-s_3)}$$

The computer should be careful to use the corrected values of A_1, A_3. (See VIII.) Trifling errors in the angles should be distributed.

$$e\sin\tfrac{1}{2}(v_3+v_1)=\frac{\frac{p}{r_1}-\frac{p}{r_3}}{2\sin\frac{1}{2}(v_3-v_1)}$$

$$e\cos\tfrac{1}{2}(v_3+v_1)=\frac{\frac{p}{r_1}+\frac{p}{r_3}-2}{2\cos\frac{1}{2}(v_3-v_1)}$$

$$\tan\tfrac{1}{2}(v_3+v_1) \qquad e^2$$

For control:

$$e\cos v_2=\frac{p}{r_2}-1$$

$$\epsilon=\sqrt{\frac{1-e}{1+e}} \qquad a=\frac{p}{1-e^2} \qquad b=\sqrt{(ap)}$$

Direction-cosines of semi-major axis.

$$l=-\frac{\cos v_2}{s_2}\alpha_2-\frac{\sin v_2}{s_5}\alpha_5$$

$$m=-\frac{\cos v_2}{s_2}\beta_2-\frac{\sin v_2}{s_5}\beta_5$$

$$n=-\frac{\cos v_2}{s_2}\gamma_2-\frac{\sin v_2}{s_5}\gamma_5$$

Direction-cosines of semi-minor axis.

$$\lambda = -\frac{\sin v_2}{s_2}a_2 + \frac{\cos v_2}{s_5}a_5$$

$$\mu = -\frac{\sin v_2}{s_2}\beta_2 + \frac{\cos v_2}{s_5}\beta_5$$

$$\nu = -\frac{\sin v_2}{s_2}\gamma_2 + \frac{\cos v_2}{s_5}\gamma_5$$

Components of the semi-axes.

$$a_x = al \qquad a_y = am \qquad a_z = an$$
$$b_x = b\lambda \qquad b_y = b\mu \qquad b_z = b\nu$$

X.

Time of perihelion passage.

Corrections for aberration.

$$\tan \tfrac{1}{2}E_1 = \epsilon \tan \tfrac{1}{2}v_1 \qquad \delta t_1 = -\cdot 0057613\rho_1$$
$$\tan \tfrac{1}{2}E_2 = \epsilon \tan \tfrac{1}{2}v_2 \qquad \delta t_2 = -\cdot 0057613\rho_2$$
$$\tan \tfrac{1}{2}E_3 = \epsilon \tan \tfrac{1}{2}v_3 \qquad \delta t_3 = -\cdot 0057613\rho_3$$

$$\log \cdot 0057613 = 7\cdot 76052$$

$$t_1 + \delta t_1 - T = k^{-1}a^{\frac{3}{2}}(E_1 - e\sin E_1)$$
$$t_2 + \delta t_2 - T = k^{-1}a^{\frac{3}{2}}(E_2 - e\sin E_2)$$
$$t_3 + \delta t_3 - T = k^{-1}a^{\frac{3}{2}}(E_3 - e\sin E_3)$$

The threefold determination of T affords a control of the exactness of the solution of the problem. If the discrepancies in the values of T are such as to require another correction of the formulæ (a third hypothesis), this may be based on the equations

$$\Delta \log \tau_1 = M\frac{T_{(3)} - T_{(2)}}{t_3 - t_2} \qquad \Delta \log \tau_3 = M\frac{T_{(2)} - T_{(1)}}{t_2 - t_1}$$

where $T_{(1)}$, $T_{(2)}$, $T_{(3)}$ denote respectively the values obtained from the first, second, and third observations, and M the modulus of common logarithms.

XI.

For an ephemeris.

$$\frac{k}{a^{\frac{3}{2}}}(t - T) = E - e\sin E$$

Heliocentric coordinates. (Components of $\mathfrak{R}$.)

$$x = -ea_x + a_x\cos E + b_x\sin E$$
$$y = -ea_y + a_y\cos E + b_y\sin E$$
$$z = -ea_z + a_z\cos E + b_z\sin E$$

These equations are completely controlled by the agreement of the computed and observed positions and the following relations between the constants:

$$a_xb_x + a_yb_y + a_zb_z = 0 \qquad a_x^2 + a_y^2 + a_z^2 = a^2 \qquad b_x^2 + b_y^2 + b_z^2 = (1 - e^2)a^2$$

XII.

When an approximate orbit is known in advance, we may use it to improve our fundamental equation. The following appears to be the most simple method:

Find the excentric anomalies E_1, E_2, E_3, and the heliocentric distances r_1, r_2, r_3, which belong in the approximate orbit to the times of observation corrected for aberration.

Calculate B_1, B_3, as in § I, using these corrected times.

Determine A_1, A_3 by the equation

$$\frac{A_1\left(1+\frac{B_1}{r_1^3}\right)}{\sin(E_3-E_2)-e\sin E_3+e\sin E_2}=\frac{A_3\left(1+\frac{B_3}{r_3^3}\right)}{\sin(E_2-E_1)-e\sin E_2+e\sin E_1}$$

in connection with the relation $A_1+A_3=1$.

Determine B_2 so as to make

$$\frac{A_1\frac{B_1}{r_1^3}+\frac{B_2}{r_2^3}+A_3\frac{B_3}{r_3^3}}{4\sin\tfrac{1}{2}(E_2-E_1)\sin\tfrac{1}{2}(E_3-E_2)\sin\tfrac{1}{2}(E_3-E_1)}$$

equal to either member of the last equation.

It is not necessary that the times for which E_1, E_2, E_3, r_1, r_2, r_3, are calculated should precisely agree with the times of observation corrected for aberration. Let the former be represented by t_1', t_2', t_3', and the latter by t_1'', t_2'', t_3''; and let

$$\Delta\log\tau_1=\log(t_3''-t_2'')-\log(t_3'-t_2'),$$
$$\Delta\log\tau_3=\log(t_2''-t_1'')-\log(t_2'-t_1').$$

We may find B_1, B_3, A_1, A_3, B_2, as above, using t_1', t_2', t_3', and then use $\Delta\log\tau_1$, $\Delta\log\tau_3$ to correct their values, as in § VIII.

Numerical Example.

To illustrate the numerical computations we have chosen the following example, both on account of the large heliocentric motion, and because Gauss and Oppolzer have treated the same data by their different methods.

The data are taken from the *Theoria Motus*, § 159, viz.,

Times, 1805, September -	5·51336	139·42711	265·39813
Longitudes of Ceres - -	95° 32′ 18″·56	99° 49′ 5″·87	118° 5′ 28″·85
Latitudes of Ceres - -	−0° 59′ 34″·06	+7° 16′ 36″·80	+7° 38′ 49″·39
Longitudes of the Earth -	342° 54′ 56″·00	117° 12′ 43″·25	241° 58′ 50″·71
Logs of the Sun's distance -	0·0031514	9·9929861	0·0056974

The positions of Ceres have been freed from the effects of parallax and aberration.

I.

From the given times we obtain the following values:

	Numbers.	Logarithms.
$t_2 - t_1$	133·91375	2·1268252
$t_3 - t_2$	125·97102	2·1002706
$t_3 - t_1$	259·88477	2·4147809
A_1	·4847187	9·6854897
A_3	·5152812	9·7120443
τ_1		·3358520
τ_3		·3624066
B_1		9·6692113
B_2		·3183722
B_3		9·5623916

Control:

$$A_1B_1 + B_2 + A_3B_3 = 2{\cdot}4959086$$
$$\tfrac{1}{2}\tau_1\tau_3 = 2{\cdot}4959081$$

II.

From the given positions we get:

$\log X_1$	9·9835515	+	$\log X_2$	9·6531725	−	$\log X_3$	9·6775810	−
$\log Y_1$	9·4711748	−	$\log Y_2$	9·9420444	+	$\log Y_3$	9·9515547	−
Z_1	0		Z_2	0		Z_3	0	
$\log \xi_1$	8·9845270	−	$\log \xi_2$	9·2282738	−	$\log \xi_3$	9·6690294	−
$\log \eta_1$	9·9979027	+	$\log \eta_2$	9·9900800	+	$\log \eta_3$	9·9416855	+
$\log \zeta_1$	8·2387150	−	$\log \zeta_2$	9·1026549	+	$\log \zeta_3$	9·1240813	+
$\mathfrak{E}_1 \cdot \mathfrak{F}_1$	·3874081	−	$\mathfrak{E}_2 \cdot \mathfrak{F}_2$	·9314223	+	$\mathfrak{E}_3 \cdot \mathfrak{F}_3$	·5599304	−
p_1^2	·8645336	+	p_2^2	·1006681	+	p_3^2	·7130624	+

III.

The preceding computations furnish the numerical values for the equations III_1, III', III_2, III'', III_3, III''', which follow. Brackets indicate that logarithms have been substituted for numbers.

We have now to assume some values for the heliocentric distances r_1, r_2, r_3. A mean proportional between the mean distances of Mars and Jupiter from the Sun suggests itself as a reasonable assumption. In order, however, to test the convergence of the computations, when the assumptions are not happy, we will make the much less probable assumption (actually much farther from the truth) that the heliocentric distances are an arithmetical mean between the distances of Mars and Jupiter. This gives ·526 for the logarithm of each of the distances r_1, r_2, r_3. From these assumed values we compute the first column of numbers in the three following tables:

$$\left.\begin{array}{ll} q_1=\rho_1-\cdot3874081 & \alpha_1=-[8\cdot6700167](q_1-9\cdot5901555)(1+R_1) \\ r_1^{\,2}=q_1^{\,2}+\cdot8645336 & \beta_1=\;\;[9\cdot6833924](q_1+\;\cdot0900552)(1+R_1) \\ R_1=[9\cdot6692113]r_1^{\,-3} & \gamma_1=-[7\cdot9242047](q_1+\;\cdot3874081)(1+R_1) \end{array}\right\}\text{III}_1$$

$$P'=\frac{[\cdot47712]R_1q_1}{(1+R_1)r_1^{\,2}} \qquad \left.\begin{array}{l} \alpha'=-\cdot046775-[8\cdot67002]R_1-P'\alpha_1 \\ \beta'=\;\;\cdot482383+[9\cdot68339]R_1-P'\beta_1 \\ \gamma'=-\cdot008399-[7\cdot92420]R_1-P'\gamma_1 \end{array}\right\}\text{III}'$$

Δq_1			−·66731	−·04558	−·0010434	+·0000006
q_1	+	3·22606	2·55875	2·51317	2·5142134	2·5142140
$\log r_1$	+	·52600	·434960	·4280791	·4282376	·4282377
$\log R_1$	+	8·09121	8·364331	8·3849740	8·3844985	
$\log(1+R_1)$	+	·00533	·009934	·0104122	·0104010	
$\log P'$	+	8·01967	8·369626	8·3957468	8·3951457	
α_1	+	·30136	·336506	·3390605	·3390018	
β_1	+	1·61938	1·307304	1·286223	1·2867056	
γ_1	−	·03072	·025316	·0249518	·0249601	
α'	−	·050505		·0563438		
β'	+	·47139		·4620942		
γ'	−	·00818		·0079821		

$$\left.\begin{array}{ll} q_2=\rho_2+\cdot9314223 & \alpha_2=+[9\cdot2282738](q_2+1\cdot7286820)(1-R_2) \\ r_2^{\,2}=q_2^{\,2}+\cdot1006681 & \beta_2=-[9\cdot9900800](q_2-\;\cdot0361309)(1-R_2) \\ R_2=[0\cdot3183722]r_2^{\,-3} & \gamma_2=-[9\cdot1026549](q_2-\;\cdot9314223)(1-R_2) \end{array}\right\}\text{III}_2$$

$$P''=\frac{[\cdot47712]R_2q_2}{(1-R_2)r_2^{\,2}} \qquad \left.\begin{array}{l} \alpha''=\;\;\cdot169151-[9\cdot22827]R_2+P''\alpha_2 \\ \beta''=-\cdot977417+[9\cdot99008]R_2+P''\beta_2 \\ \gamma''=-\cdot126664+[9\cdot10265]R_2+P''\gamma_2 \end{array}\right\}\text{III}''$$

Δq_2			−·77826	+·005042	+·0013222	+·0000021
q_2	+	3·34235	2·56409	2·569132	2·5704542	2·5704563
$\log r_2$	+	·52600	·412233	·4130733	·4132934	·4132937
$\log R_2$	+	8·74037	9·081673	9·0791524	9·0784920	
$\log(1-R_2)$	+	9·97543	9·944142	9·9444866	9·9445766	
$\log P''$	+	8·71411	9·199120	9·1954270	9·1944598	
α_2	+	·81059	·638489	·6397466	·6400760	
β_2	−	3·05379	2·172660	2·1787230	2·1803116	
γ_2	−	·28858	·181843	·1825486	·1827338	
α''	+	·20182		·2491854		
β''	−	1·08177		1·2018221		
γ''	−	·13464		·1400944		

$$\left.\begin{array}{ll} q_3=\rho_3-\cdot5599304 & \alpha_3=-[9\cdot3810737](q_3+1\cdot5798163)(1+R_3) \\ r_3^{\,2}=q_3^{\,2}+\cdot7130624 & \beta_3=\;\;[9\cdot6537308](q_3-\;\cdot4630521)(1+R_3) \\ R_3=[9\cdot5623916]r_3^{\,-3} & \gamma_3=\;\;[8\cdot8361256](q_3+\;\cdot5599304)(1+R_3) \end{array}\right\}\text{III}_3$$

$$P'''=\frac{[\cdot47712]R_3q_3}{(1+R_3)r_3^{\,2}} \qquad \left.\begin{array}{l} \alpha'''=-\cdot240477-[9\cdot38107]R_3-P'''\alpha_3 \\ \beta'''=+\cdot450537+[9\cdot65373]R_3-P'''\beta_3 \\ \gamma'''=+\cdot068569+[8\cdot83613]R_3-P'''\gamma_3 \end{array}\right\}\text{III}'''$$

Δq_3			−·80780	−·04055	+·0025316	+·0000031
q_3	+	3·24945	2·44165	2·40110	2·4036316	2·4036347
$\log r_3$	+	0·52600	·412217	·4057319	·4061394	·4061399
$\log R_3$	+	7·98439	8·325742	8·3451948	8·3439733	
$\log(1+R_3)$	+	·00417	·009099	·0095108	·0094843	
$\log P'''$	+	7·91715	8·357016	8·3817516	8·3801993	
a_3	−	1·17253	·987590	·9785152	·9790776	
β_3	+	1·26749	·910305	·8924956	·8936069	
γ_3	+	·26373	·210171	·2075292	·2076940	
α'''	−	·22847		·2222335		
β'''	+	·44441		·4390163		
γ'''	+	·06690		·0650888		

IV.

The values of α', β', etc., furnish the basis for the computation of the following quantities:

$$a_1 = -·01254 \qquad a_2 = -·03517 \qquad a_3 = -·07232$$
$$b_1 = +·01726 \qquad b_2 = -·00525 \qquad b_3 = -·00845$$
$$c_1 = -·15746 \qquad c_2 = -·08526 \qquad c_3 = -·04050$$

For G we get three values sensibly identical. Adopting the mean, we set

$$G = ·01006.$$

We also get $H = -·00998, \qquad L = ·02322.$*

V.

Taking the values of a_1, a_2, etc., from the columns under III_1, III_2, III_3, we form the residuals

$$\alpha = -·06058, \qquad \beta = -·16692, \qquad \gamma = -·05557.$$

From these, with the numbers last computed, we get

$$C_1 = -·65888, \qquad C_2 = -·76983, \qquad C_3 = -·79939,$$

which might be used as corrections for our values of q_1, q_2, q_3. To get more accurate values for these corrections we set

$$\Delta q_2 = C_2 - \tfrac{6}{10} L(\Delta q_2)^2, \quad \text{or} \quad \Delta q_2 = -·76983 - ·01393(\Delta q_2)^2,$$

which gives $\Delta q_2 = -·77826.$

The quadratic term diminishes the value of Δq_2 by ·00843. Subtracting the same quantity from C_1 and C_2 we get

$$\Delta q_1 = -·66731, \qquad \Delta q_3 = -·80780.$$

* It would have been better to omit altogether the calculation of H and L, if the small value of the latter could have been foreseen. In fact, it will be found that the terms containing L hardly improve the convergence, being smaller than quantities which have been neglected. Nevertheless, the use of these terms in this example will illustrate a process which in other cases may be beneficial.

VI.

Applying these corrections to the values of q_1, q_2, q_3 we compute the second numerical columns under equations III_1, III_2, and III_3. We do not go on to the computations by equations III′, etc., but content ourselves with the old values of a_1, b_1, etc., G, and L, which with the new residuals

$$\alpha = -\cdot 012595, \qquad \beta = \cdot 044949, \qquad \gamma = \cdot 003012,$$

give $C_1 = -\cdot 04567, \qquad C_2 = \cdot 004952, \qquad C_3 = -\cdot 04064.$

$$\Delta q_2 = C_2 - L(Dq_2 + \tfrac{1}{2}C_2)\Delta q_2 = \cdot 004952 - \cdot 02322(-\cdot 77826 + \cdot 00247)\Delta q_2.$$

This gives $\Delta q_2 = \cdot 005042.$

As the term containing L has increased the value of Δq_2 by ·00009, we add this quantity to C_1 and C_3, and get

$$\Delta q_1 = -\cdot 04558, \qquad \Delta q_3 = -\cdot 04055.$$

With these corrections we compute the third numerical columns under equations III_1, etc. This time we recompute the quantities α', etc., with which we repeat the principal computations of IV, and get the new values

$$a_1 = -\cdot 0167215 \qquad a_2 = -\cdot 0335815 \qquad a_3 = -\cdot 0743299$$
$$b_1 = +\cdot 0149145 \qquad b_2 = -\cdot 0054413 \qquad b_3 = -\cdot 0098825$$
$$c_1 = -\cdot 1576886 \qquad c_2 = -\cdot 0779570 \qquad c_3 = -\cdot 0474318$$
$$G = \cdot 0090929.$$

The quantities H and L we neglect as of no consequence at this stage of the approximation.

With these values the new residuals

$$\alpha = +\cdot 0002919, \qquad \beta = -\cdot 0000044, \qquad \gamma = +\cdot 0000288,$$

give $\Delta q_1 = C_1 = +\cdot 0010434, \qquad \Delta q_2 = C_2 = +\cdot 0013222,$

$$\Delta q_3 = C_3 = +\cdot 0025316.$$

These corrections furnish the basis for the fourth columns of numbers under equations III_1, etc., which give the residuals

$$\alpha = +\cdot 0000002, \qquad \beta = +\cdot 0000009, \qquad \gamma = +\cdot 0000001,$$

and the new corrections

$$\Delta q_1 = +\cdot 0000006, \qquad \Delta q_2 = +\cdot 0000021, \qquad \Delta q_3 = +\cdot 0000031.$$

The corrected values of q_1, q_2, q_3 give

$$\log r_1 = 0\cdot 4282377, \quad \log r_2 = 0\cdot 4132937, \quad \log r_3 = 0\cdot 4061399.$$

We have carried the approximation farther than is necessary for the following *correction of the formula*, in order to see exactly where the uncorrected formula would lead us, and for the control afforded by the fourth residuals.

VII.

The computations for the test of the uncorrected formula (the first hypothesis) are as follows:

		Number or arc.	Logarithm.
r_1			0·4282377
r_2			0·4132937
r_3			0·4061399
$A_1B_1r_1^{-3}$	+	·01174865	8·0699879
$B_2r_2^{-3}$	+	·11980944	9·0784911
$A_3B_3r_3^{-3}$	+	·01137670	8·0560162
N	+	·14293479	9·1551380
s_1	+	1·3308476	0·1241283
s_2	+	2·2796616	0·3578704
s_3	+	1·3417404	0·1276685
s	+	2·4761248	0·3937725
$s-s_1$	+	1·1452772	0·0589106
$s-s_2$	+	0·1964632	9·2932812
$s-s_3$	+	1·1343844	0·0547602
R	+		9·5065898
p	+		0·4391732
$\tan\frac{1}{2}(v_2-v_1)$	+	15° 48′ 10″·82	9·4518296
$\tan\frac{1}{2}(v_3-v_2)$	+	15° 39′ 36″·38	9·4476792
$\tan\frac{1}{2}(v_3-v_1)$	+	31° 27′ 47″·20	9·7866915
$e\sin\frac{1}{2}(v_3+v_1)$	−		8·7099387
$e\cos\frac{1}{2}(v_3+v_1)$	+		8·7872701
$\tan\frac{1}{2}(v_3+v_1)$	−	−39° 55′ 32″·31	9·9226686
e	+		8·9025438
ϵ	+		9·9652259
a	+		0·4419546
$\tan\frac{1}{2}v_1$	−	−35° 41′ 39″·75	9·8563809
$\tan\frac{1}{2}v_2$	−	−19° 53′ 28″·93	9·5584981
$\tan\frac{1}{2}v_3$	−	− 4° 13′ 52″·55	8·8691380
$\tan\frac{1}{2}E_1$	−	−33° 33′ 0″·17	9·8216068
$\tan\frac{1}{2}E_2$	−	−18° 28′ 6″·35	9·5237240
$\tan\frac{1}{2}E_3$	−	− 3° 54′ 24″·21	8·8343639
$\sin E_1$	−	−67° 6′ 0″·34	9·9643473
$\sin E_2$	−	−36° 56′ 12″·70	9·7788272
$\sin E_3$	−	− 7° 48′ 48″·42	9·1333734
$ea^{\frac{3}{2}}\sin E_1$	−	·3387061	9·5298230
$ea^{\frac{3}{2}}\sin E_2$	−	·2209545	9·3443029
$ea^{\frac{3}{2}}\sin E_3$	−	·0499861	8·6988491
$a^{\frac{3}{2}}(E_2-E_1)$	+	2·4226307	0·3842872
$a^{\frac{3}{2}}(E_3-E_2)$	+	2·3391145	0·3690515
τ_3 calc.	+	2·3048791	0·3626482
τ_1 calc.	+	2·1681461	0·3360885

VIII.

The logarithms of the calculated values of the intervals of time exceed those of the given values by ·0002416 for the first interval (τ_3) and ·0002365 for the second (τ_1). Therefore, since the corrections for aberration have been incorporated in the data, we set for the correction of the formula (for the second hypothesis)

$$\Delta \log \tau_1 = -\cdot 0002365 \qquad \Delta \log \tau_3 = -\cdot 0002416$$

This gives $\Delta \log A_1 = \cdot 0000026$ $\Delta \log A_3 = -\cdot 0000025$

$\Delta \log B_1 = -\cdot 0004872$ $\Delta \log B_2 = -\cdot 0004782$ $\Delta \log B_3 = -\cdot 0004665$

The new values of the logarithms of A_1, A_3 are

$$\log A_1 = 9\cdot 6854923 \qquad \log A_3 = 9\cdot 7120418$$

Applying these corrections to equations III_1, III_2, III_3,* we get the following:

$$\left.\begin{aligned} r_1^2 &= q_1^2 + \cdot 8645336 \\ \alpha_1 &= -[8\cdot 6700193](q_1 - 9\cdot 5901555)(1+R_1) \\ R_1 &= [9\cdot 6687241] r_1^{-3} \\ \beta_1 &= +[9\cdot 6833950](q_1 + \cdot 0900552)(1+R_1) \\ \gamma_1 &= -[7\cdot 9242073](q_1 + \cdot 3874081)(1+R_1) \end{aligned}\right\} III_1 \text{ corrected.}$$

Δq_1			+ ·0002887	− ·0000217
q_1	+	2·5142140	2·5145027	2·5144810
$\log r_1$	+	·4282377	·4282816	·4282782
$\log R_1$	+	8·3838110	8·3838793	8·3838894
$\log(1+R_1)$	+	·0103847	·0103863	·0103865
α_1	+	·3389910	·3389784	·3389796
β_1	+	1·2866654	1·2868124	1·2868024
γ_1	−	·0249593	·0249619	·0249617
$\log s_1$	+			·1241571

$$\left.\begin{aligned} r_2^2 &= q_2^2 + \cdot 1006681 \\ \alpha_2 &= +[9\cdot 2282738](q_2 + 1\cdot 7286820)(1-R_2) \\ R_2 &= [0\cdot 3178940] r_2^{-3} \\ \beta_2 &= -[9\cdot 9900800](q_2 - \cdot 0361309)(1-R_2) \\ \gamma_2 &= -[9\cdot 1026549](q_2 - \cdot 9314223)(1-R_2) \end{aligned}\right\} III_2 \text{ corrected.}$$

Δq_2			− ·0000955	+ ·0000187
q_2	+	2·5704563	2·5703608	2·5703795
$\log r_2$	+	0·4132937	·4132778	·4132809
$\log R_2$	+	9·0780129	9·0780605	9·0780513
$\log(1-R_2)$	+	9·9446418	9·9446353	9·9446365
α_2	+	·6401725	·6401487	·6401532
β_2	−	2·1806412	2·1805261	2·1805482
γ_2	−	·1827615	·1827481	·1827507
$\log s_2$	+			·3579174

* The corrections may be made without rewriting the equations.

$$\left.\begin{aligned} r_3^2 &= q_3^2 + \cdot 7130624 \\ R_3 &= [9\cdot 5619251] r_3^{-3} \end{aligned}\right\} \quad \left.\begin{aligned} \alpha_3 &= -[9\cdot 3810712](q_3 + 1\cdot 5798163)(1+R_3) \\ \beta_3 &= +[9\cdot 6537283](q_3 - \cdot 4630521)(1+R_3) \\ \gamma_3 &= +[8\cdot 8361231](q_3 + \cdot 5599304)(1+R_3) \end{aligned}\right\} \text{III}_3 \text{ corrected.}$$

Δq_3			+ ·0003302	+ ·0000424
q_3	+	2·4036347	2·4039649	2·4040073
$\log r_3$	+	·4061399	·4061929	·4061998
$\log R_3$	+	8·3435055	8·3433463	8·3433257
$\log(1+R_3)$	+	·0094742	·0094708	·0094704
α_3	−	·9790500	·9791236	·9791329
β_3	+	·8935824	·8937277	·8937461
γ_3	+	·2076882	·2077097	·2077124
$\log \varepsilon_3$	+			·1277120

With these corrected equations the last values of q_1, q_2, q_3 give the residuals

$$\alpha = \cdot 0001135 \qquad \beta = -\cdot 0003934 \qquad \gamma = -\cdot 0000326$$

These give the corrections

$$\Delta q_1 = \cdot 0002887 \qquad \Delta q_2 = -\cdot 0000955 \qquad \Delta q_3 = \cdot 0003302$$

The next residuals are

$$\alpha = \cdot 0000035 \qquad \beta = \cdot 0000140 \qquad \gamma = -\cdot 0000003$$

which give the corrections

$$\Delta q_1 = -\cdot 0000217 \qquad \Delta q_2 = \cdot 0000187 \qquad \Delta q_3 = \cdot 0000424$$

The next residuals are

$$\alpha = -\cdot 0000001 \qquad \beta = \cdot 0000003 \qquad \gamma = \cdot 0000000$$

which must be regarded as entirely insensible.

IX, X.

It remains to determine the ellipse which passes through the points to which the numbers relate in the last columns under the corrected equations III_1, III_2, III_3, and also the time of perihelion passage. The computations are as follows:

		Number or arc.	Logarithm.			Number or arc.	Logarithm.
a_4	−	1·3181125	0·1199525	e	+	·08076104	8·9072019
β_4	−	·3930563	9·5944547	$\tan \frac{1}{2}v_1$	−	− 35° 26′ 25″·87	9·8523140
γ_4	+	·2326741	9·3667481	$\tan \frac{1}{2}v_2$	−	− 19° 38′ 18″·20	9·5524725
s_4^2	+	1·9460513	0·2891543	$\tan \frac{1}{2}v_3$	−	− 3° 58′ 47″·06	8·8424313
s_4	+	1·3950100	0·1445772	ϵ	+		9·9648493
s_1	+	1·3309358		a	+		0·4424586
s_3	+	1·3418749		b	+		0·4410376
s_2	+	2·2799084	0·3579174	l	−	·8139130	9·9105780
s	+	2·4763596	0·3938137	m	+	·5564152	9·7453990
$s - s_1$	+	1·1454238	0·0589662	n	+	·1671768	9·2231760
$s - s_3$	+	1·1344847	0·0547986	λ	−	·5517065	9·7417081
S	+	2·0339104	0·3083318	μ	−	·8304016	9·9192881
$S - s_1$	+	·7029746	9·8469397	ν	+	·07780215	8·8909916
$S - s_4$	+	·6389004	9·8054332	a_x	−		0·3530366
$S - s_3$	+	·6920355	9·8401284	a_y	+		0·1878576
$\mathbb{S}_4 . \mathbb{S}_2$*	−	·0292373	8·4659373	a_z	+		9·6656346
$\mathbb{S}_4 . \mathbb{S}_2 s_2^{-2}$	−		7·7501025	b_x	−		0·1827457
$\mathbb{S}_4 . \mathbb{S}_2 s_2^{-2} a_2$	−	·003600696	7·5563864	b_y	−		0·3603257
$\mathbb{S}_4 . \mathbb{S}_2 s_2^{-2} \beta_2$	+	·012265017	8·0886682	b_z	+		9·3320292
$\mathbb{S}_4 . \mathbb{S}_2 s_2^{-2} \gamma_2$	+	·001027926	7·0119616	$\tan \frac{1}{2}E_1$	−	− 33° 16′ 50″·08	9·8171633
a_5	−	1·3145118	0·1187645	$\tan \frac{1}{2}E_2$	−	− 18° 12′ 58″·10	9·5173218
β_5	−	·4053213	9·6077994	$\tan \frac{1}{2}E_3$	−	− 3° 40′ 16″·33	8·8072806
γ_5	+	·2316462	9·3648252	$k^{-1} a^{\frac{3}{2}} e \sin E_1$	−	19·85671	1·2979074
s_5^2	+	1·9458866	0·2891176	$k^{-1} a^{\frac{3}{2}} e \sin E_2$	−	12·85291	1·1090014
R^2	+		9·0132057	$k^{-1} a^{\frac{3}{2}} e \sin E_3$	−	2·76589	·4418347
R	+		9·5066028	$k^{-1} a^{\frac{3}{2}} E_1$	−	311·31839	2·4932048
N	+	·14278017	9·1546679	$k^{-1} a^{\frac{3}{2}} E_2$	−	170·40027	2·2314703
p	+	2·7517987	0·4396167	$k^{-1} a^{\frac{3}{2}} E_3$	−	34·34174	1·5358223
$\tan \frac{1}{2}(v_2 - v_1)$	+	15° 48′ 7″·67	9·4518042	$T_{(1)}$	+	296·97504	
$\tan \frac{1}{2}(v_3 - v_2)$	+	15° 39′ 31″·14	9·4476366	$T_{(2)}$	+	296·97447	
$\tan \frac{1}{2}(v_3 - v_1)$	+	31° 27′ 38″·81	9·7866517	$T_{(3)}$	+	296·97398	
$e \sin \frac{1}{2}(v_3 + v_1)$	−		8·7099782	T	+	296·97450	
$e \cos \frac{1}{2}(v_3 + v_1)$	+		8·7951055				
$\tan \frac{1}{2}(v_3 + v_1)$	−	39° 25′ 12″·94	9·9148727				

* $\mathbb{S}_4 . \mathbb{S}_2 = \alpha_4 \alpha_2 + \beta_4 \beta_2 + \gamma_4 \gamma_2$.

XI.

This gives the following equations for an ephemeris:

$T = 1806$, June, 23·97450, Paris mean time

$$[2{\cdot}8863186](t-T) = E_{\text{in seconds}} - [4{\cdot}2216270]\sin E$$

Heliocentric coordinates relating to the ecliptic.

$$x = +{\cdot}1820700 - [0{\cdot}3530366]\cos E - [0{\cdot}1827457]\sin E$$
$$y = -{\cdot}1244685 + [0{\cdot}1878576]\cos E - [0{\cdot}3603257]\sin E$$
$$z = -{\cdot}0373970 + [9{\cdot}6656346]\cos E + [9{\cdot}3320292]\sin E$$

The differences of the values of $T_{(1)}$, $T_{(2)}$, $T_{(3)}$, from their mean T, indicate the residual errors of this hypothesis. They indicate differences in the calculated and the observed geocentric positions which are represented by the geocentric angles subtended by the path described by the planet in the following fractions of a day: ·00054, ·00003, ·00052. Since the heliocentric motion of the planet is about one-fourth of a degree per day, and the planet is considerably farther from the earth than from the sun at the times of the first and third observations, the errors will be less than half a second in arc.

If we desire all the accuracy possible with seven-figure logarithms, we may form a *third hypothesis* based on the following corrections:

$$\Delta \log \tau_1 = M\frac{T_{(3)} - T_{(2)}}{t_3 - t_2} = -{\cdot}0000017,$$

$$\Delta \log \tau_3 = M\frac{T_{(2)} - T_{(1)}}{t_2 - t_1} = -{\cdot}0000018.$$

The equations for an ephemeris will then be:

$T = 1806$, June 23·96378, Paris mean time

$$[2{\cdot}8863140](t-T) = E_{\text{in seconds}} - [4{\cdot}2216530]\sin E$$

Heliocentric coordinates relating to the ecliptic.

$$x = +{\cdot}1820765 - [0{\cdot}3530261]\cos E - [0{\cdot}1827783]\sin E$$
$$y = -{\cdot}1244853 + [0{\cdot}1878904]\cos E - [0{\cdot}3603153]\sin E$$
$$z = -{\cdot}0373987 + [9{\cdot}6656285]\cos E + [9{\cdot}3320758]\sin E$$

The agreement of the calculated geocentric positions with the data is shown in the following table:

Times, 1805, September	5·51336	139·42711	265·39813
Second hypothesis:			
longitudes - -	95° 32′ 18″·88	99° 49′ 5″·87	118° 5′ 28″·52
errors - - -	0″·32	0″·00	−0″·33
latitudes - -	−0° 59′ 34″·01	7° 16′ 36″·82	7° 38′ 49″·34
errors - - -	0″·05	0″·02	−0″·05
Third hypothesis:			
longitudes - -	95° 32′ 18″·65	99° 49′ 5″·82	118° 5′ 28″·79
errors - - -	0″·09	−0″·05	−0″·06
latitudes - -	−0° 59′ 34″·04	7° 16′ 36″·78	7° 38′ 49″·38
errors - - -	0″·02	−0″·02	−0″·01

The immediate result of each hypothesis is to give three positions of the planet, from which, with the times, the orbit may be calculated in various ways, and with different results, so far as the positions deviate from the truth on account of the approximate nature of the hypothesis. In some respects, therefore, the correctness of an hypothesis is best shown by the values of the geocentric or heliocentric distances which are derived directly from it. The logarithms of the heliocentric distances are brought together in the following table, and corresponding values from Gauss* and Oppolzer† are added for comparison. It is worthy of notice that the positions given by our second hypothesis are substantially correct, and if the orbit had been calculated from the first and third of these positions with the interval of time, it would have left little to be desired.

	$\log r_1$.	$\log r_2$.	$\log r_3$.
First hypothesis - -	·4282377	·4132937	·4061399
Second hypothesis - -	·4282782	·4132809	·4061998
Third hypothesis - -	·4282786	·4132808	·4062003
Gauss :			
First hypothesis -	·4323934	·4114726	·4094712
Second hypothesis -	·4291773	·4129371	·4071975
Third hypothesis -	·4284841	·4132107	·4064697
Fourth hypothesis -	·4282792	·4132817	·4062033
Oppolzer :			
First hypothesis -	·4281340	·413330	·4061699
Second hypothesis -	·4282794	·4132801	·4061976
Third hypothesis -	·4282787		·4062009

In comparing the different methods, it should be observed that the determination of the positions in any hypothesis by Gauss's method requires successive corrections of a single independent variable, a corresponding determination by Oppolzer's method requires the successive corrections of two independent variables, while the corresponding determination by the method of the present paper requires the successive corrections of three independent variables.

* *Theoria motus*, § 159.

† *Lehrbuch zur Bahnbestimmung der Kometen und Planeten*, 2nd ed., vol. i, p. 394.

VI.

ON THE USE OF THE VECTOR METHOD IN THE DETERMINATION OF ORBITS

LETTER TO DR. HUGO BUCHHOLZ, EDITOR OF KLINKERFUES' *Theoretische Astronomie.**

New Haven, *October*, 1898.

Dr. HUGO BUCHHOLZ,

My dear Sir,—The opinion of Fabritius† on the comparative convenience of different methods is entitled to far more weight than mine, for I am no astronomer, and have calculated very few orbits, none, indeed, except for the trial of my own formulæ. The object of my paper was to show to astronomers, who are rather conservative (and with right, for astronomy is the oldest of the exact sciences), the advantage in the use of vector notations, which I had learned in Physics from Maxwell. This object could be best obtained, not by showing, as I might have done, that much in the classic methods could be conveniently and perspicuously represented by vector notations, but rather by showing that these notations so simplify the subject, that it is easy to construct a method for the complete solution of the problem. That the method given is the best possible, I certainly do not claim, but only that it is much better than I could have found without the use of vector notations. Some of the more obvious crudities in my paper have been corrected in that of Beebe and Phillips.‡ Doubtless many more remain, even if the general method be preserved.

My first efforts, however, to solve the fundamental approximative equation were along the same lines which Fabritius has followed:—to set $r_1 = r_2$ and $r_3 = r_2$ in equation second of (2) of Fabritius, which will give ρ_2 and r_2, then to get r_1 from the first of (3) of Fabritius, and then r_3 either from equation second of (3) or from some other

* [In which the preceding memoir, and also that of Beebe and Phillips referred to below, are translated.]

† [See Fabritius, W. "Ueber eine leichte Methode der Bahnbestimmung mit Zugrundelegung des Princips von Gibbs," also "Weitere Anwendungen des Gibbs'chen Princips," *Astronomische Nachrichten*, No. 3061, 3065 (1891)].

‡ ["The Orbit of Swift's comet, 1880 V, Determined by Gibbs's Vector Method," W. Beebe and A. W. Phillips. *Gould's Astronomical Journal*, vol. ix, Dec. 1889.]

which would serve the purpose, and then to find better values of ρ_2, r_2 by setting in equation second of (2)

$$r_1=\left[\frac{r_1}{r_2}\right]r_2, \quad r_3=\left[\frac{r_3}{r_2}\right]r_2,$$

the expressions in brackets denoting numbers derived from the approximate values already found. This is similar to or identical with the method of Fabritius, except that he combines with it the principle of interpolation (for the first value in the third "hypothesis"). As I found the approximation by this method sometimes slow or failing, notably in the case of Swift's comet, 1880 V, I tried the method published in my paper. Indeed, it may be said that the method of my paper was constructed to meet the exigencies of the case of the comet, 1880 V.

In ordinary cases I think that the method of Fabritius may very likely be better than that which I published. The equations are very simply and perspicuously represented in vector notations. I shall use the notations of my paper, writing $\overline{E}$, $\overline{F}$, etc., for German letters.* To eliminate ρ_1 and ρ_3 from equation (7) in my paper, multiply *directly* by $\mathfrak{F}_1\times\mathfrak{F}_3$. This gives

$$A_1\left(1+\frac{B_1}{r_1^3}\right)(\mathfrak{E}_1\mathfrak{F}_1\mathfrak{F}_3)-\left(1-\frac{B_2}{r_2^3}\right)[(\mathfrak{E}_2\mathfrak{F}_1\mathfrak{F}_3)+\rho_2(\mathfrak{F}_2\mathfrak{F}_1\mathfrak{F}_3)]$$
$$+A_3\left(1+\frac{B_3}{r_3^3}\right)(\mathfrak{E}_3\mathfrak{F}_1\mathfrak{F}_3)=0. \qquad (a)$$

To eliminate ρ_3 and r_3, multiply by $\mathfrak{E}_3\times\mathfrak{F}_3$ which gives

$$A_1\left(1+\frac{B_1}{r_1^3}\right)[(\mathfrak{E}_1\mathfrak{E}_3\mathfrak{F}_3)+\rho_1(\mathfrak{F}_1\mathfrak{E}_3\mathfrak{F}_3)]$$
$$-\left(1-\frac{B_2}{r_2^3}\right)[(\mathfrak{E}_1\mathfrak{E}_3\mathfrak{F}_3)+\rho_2(\mathfrak{F}_2\mathfrak{E}_3\mathfrak{F}_3)]=0. \qquad (b)$$

When we have found ρ_1, r_1, ρ_2, r_2 it is not necessary to eliminate any of them, and to save labor in forming the equation for ρ_3, r_3, I should be inclined to take the components in (7) in the direction of one of the coordinate axes, choosing that one which is most nearly directed towards the third observed position. However, I will write

$$A_1\left(1+\frac{B_1}{r_1^3}\right)[(\mathfrak{E}_1.\mathfrak{P})+\rho_1(\mathfrak{F}_1.\mathfrak{P})]-\left(1-\frac{B_2}{r_2^3}\right)[(\mathfrak{E}_2.\mathfrak{P})+\rho_2(\mathfrak{F}_2.\mathfrak{P})]$$
$$+A_3\left(1+\frac{B_3}{r_3^3}\right)[(\mathfrak{E}_3.\mathfrak{P})+\rho_3(\mathfrak{F}_3.\mathfrak{P})]=0, \qquad (c)$$

where $\mathfrak{P}$ may represent an axis of coordinates, or $(\mathfrak{E}_1\times\mathfrak{F}_1)$ which would give Fabritius' equation. It might be directed towards the pole

*[In the remainder of the letter as here printed German capitals have been substituted for the $\overline{E}$, $\overline{F}$, $\overline{P}$, etc. of the original, thus making the notation uniform with that of the paper referred to.]

of the ecliptic, which would make $(\mathfrak{E}_1.\mathfrak{P})$, $(\mathfrak{E}_2.\mathfrak{P})$, $(\mathfrak{E}_3.\mathfrak{P})$ vanish, except for exceedingly minute quantities depending on the latitude of the sun and the geocentric coordinates of the observatories, if these are included in $\mathfrak{E}_1$, $\mathfrak{E}_2$, $\mathfrak{E}_3$.

The equations (a), (b), (c), which are together equivalent to (7), I would solve as follows, almost in the same way as Fabritius, but relying a little more on *interpolation*, and less on the convergence of which he speaks, which in special cases may more or less fail.

Setting $r_1 = r_2$ and $r_3 = r_2$ in (a), which thus modified I shall call (a'), and solving this (a') by "trial and error," using ρ_2 as the independent variable, as soon as I have a value of ρ_2 which I think will give a residual of (a') of the same order of smallness as the effect of changing $\frac{1}{r_1^3}$ and $\frac{1}{r_3^3}$ into $\frac{1}{r_2^3}$, I determine from this value by (b) and (c), r_1 and r_3, and then find the residual of (a), using the values of r_1, r_2, r_3 derived all from the same assumed ρ_2. Now using the last value of $\frac{\Delta \text{ (residual)}}{\Delta \rho_2}$ in my previous calculations on (a') which indeed applies only roughly to the (a), I would get a value ρ_2 which I would use for the second "hypothesis" in (a). This will give a second residual in (a), which will enable me to make a more satisfactory interpolation. As many more interpolations may be made as shall be found necessary.

Some such method, which should perhaps be called the method of Fabritius, would, I think, in most cases probably be the best for solution of equation (7).

Of course I am quite aware that the merit of my paper, if any, lies principally in the fundamental approximation (1). I will add a few words on this subject.

The equation may be written more symmetrically

$$\frac{\tau_1}{\tau_2}\left(1+\frac{\tau_2^2+\tau_3^2-3\tau_1^2}{24r_1^3}\right)\mathfrak{R}_1-\left(1+\frac{\tau_3^2+\tau_1^2-3\tau_2^2}{24r_2^3}\right)\mathfrak{R}_2 + \frac{\tau_3}{\tau_2}\left(1+\frac{\tau_1^2+\tau_2^2-3\tau_3^2}{24r_3^3}\right)\mathfrak{R}_3=0. \qquad \text{I}$$

It might be made entirely symmetrical by writing $-\tau_2$ for τ_2.

If an expression ending with t^3 had been used, we could still have satisfied two of the conditions relating to acceleration, and should have obtained

$$\frac{\tau_1}{\tau_2}\mathfrak{R}_1-\left(1+\frac{\tau_1^2-\tau_2^2}{6r_2^3}\right)\mathfrak{R}_2+\frac{\tau_3}{\tau_2}\left(1+\frac{\tau_1^2-\tau_3^2}{6r_3^3}\right)\mathfrak{R}_3=0, \qquad \text{II}a$$

or

$$\frac{\tau_1}{\tau_2}\left(1+\frac{\tau_2^2-\tau_1^2}{6r_1^3}\right)\mathfrak{R}_1-\mathfrak{R}_2+\frac{\tau_3}{\tau_2}\left(1+\frac{\tau_2^2-\tau_3^2}{6r_3^3}\right)\mathfrak{R}_3=0, \qquad \text{II}b$$

or

$$\frac{\tau_1}{\tau_2}\left(1+\frac{\tau_3^2-\tau_1^2}{6r_1^3}\right)\mathfrak{R}_1-\left(1+\frac{\tau_3^2-\tau_2^2}{6r_2^3}\right)\mathfrak{R}_2+\frac{\tau_3}{\tau_2}\mathfrak{R}_3=0. \qquad \text{II}c$$

Using an expansion ending with t^2 we can only satisfy one condition relating to acceleration, say the second. This will give

$$\frac{\tau_1}{\tau_2}\mathfrak{R}_1-\left(1-\frac{\tau_1\tau_3}{2r_2^3}\right)\mathfrak{R}_2+\frac{\tau_3}{\tau_2}\mathfrak{R}_3=0. \qquad \text{III}$$

(Gauss uses virtually

$$\frac{\tau_1}{\tau_2}\mathfrak{R}_1-\frac{\mathfrak{R}_2}{1+\frac{\tau_1\tau_3}{2r_2^3}}+\frac{\tau_3}{\tau_2}\mathfrak{R}_3=0,$$

which is a little more convenient, but not, I think, generally quite so accurate.)

Writing an equation analogous to III for the earth and subtracting from (7), *Mem. Nat. Acad.*, we have

$$\frac{\tau_1}{\tau_2}\rho_1\mathfrak{F}_1-\left(1-\frac{\tau_1\tau_3}{2r_2^3}\right)\rho_2\mathfrak{F}_2+\frac{\tau_3}{\tau_2}\rho_3\mathfrak{F}_3=\frac{\tau_1\tau_3}{2}\left(\frac{1}{\mathfrak{E}_2^3}-\frac{1}{r_2^3}\right)\mathfrak{E}_2,$$

which gives, on multiplication by $\mathfrak{F}_1\times\mathfrak{F}_3$ and $\mathfrak{F}_2\times\mathfrak{E}_2$, theorems of Olbers and Lambert.

It is evident that *in general* the error in I is of the fifth order, in II*a*, II*b*, II*c* of the fourth, and in III of the third. But for equal intervals, the error in I is of the sixth order, and in III of the fourth. And when $\tau_2^2+\tau_3^2-3\tau_1^2=0$, II*a* becomes identical with I, and its error is of the fifth order.

The same is true of II*c* in the corresponding case. It follows that when the intervals are nearly as 5 : 8 we should use II*a* or II*b* instead of I. This will evidently abbreviate the solution given above as only one of the quantities r_1, r_3 is to be used.

The formulæ II*a*, II*b*, II*c* may also be obtained by the following method, which will show their relative accuracy.

The interpolation formula

$$\frac{\tau_1}{\tau_2}\frac{\mathfrak{R}_1}{r_1^3}-\frac{\mathfrak{R}_2}{r_2^3}+\frac{\tau_3}{\tau_2}\frac{\mathfrak{R}_3}{r_3^3}=0$$

has an error evidently of the second order. If we multiply by $\frac{\tau_2^2+\tau_3^2-3\tau_1^2}{24}$ and subtract from I, we get II*a*. So if we multiply by

$$\frac{\tau_3^2+\tau_1^2-3\tau_2^2}{24} \quad \text{or} \quad \frac{\tau_1^2+\tau_2^2-3\tau_3^2}{24}$$

we get II*b* or II*c*. The errors due to using one of these equations instead of I are therefore proportional to these multipliers and very unequal.

Again, in case of equal intervals, II*a* and II*c* become identical with III. There is, therefore, no reason for using II*a* or II*c* when the intervals are nearly equal. II*b* is in this case much less accurate than III.

It will be observed that all the formulæ I, IIa, IIb, IIc, III, may be expressed in the general form

$$\frac{\tau_1}{\tau_2}\left(1+\frac{B_1}{r_1^3}\right)\mathfrak{R}_1-\left(1+\frac{B_2}{r_2^3}\right)\mathfrak{R}_2+\frac{\tau_3}{\tau_2}\left(1+\frac{B_3}{r_3^3}\right)\mathfrak{R}_3=0,$$

except that the letters B_1, B_2, B_3 have different values in the different cases, some vanishing in the more simple formulæ. Moreover, if the values of B_1, B_2, B_3 have been calculated for I, the values for IIa, IIb, or IIc are found simply by subtraction of one of the numbers from the three. It is evident that IIb will hardly be useful except in special cases, as in the determination of a parabolic orbit in the failing case of Olbers' method, and then it would be a question whether it would not be better to determine the orbit from ρ_2 and ρ_3, or ρ_2 and ρ_1, using IIa or IIc.

Equations IIa and IIc are very appropriate for the determination of an elliptic orbit when the observed motion is nearly in the ecliptic, by means of four observations with intervals nearly in the ratio 5 : 8 : 5.

It is evident that the solution of (7) given above may be varied, in ways too numerous to mention, by the use of the simpler forms IIa, IIc, or III for I in the earlier stages of the work. This only involves changing the values of B_1, B_2, B_3, in (a), (b) and (c).

It is not correct to say that in my expressions for the ratios of the triangles the error is of the fifth order in general, or for equal intervals, of the sixth. If we write p_1, p_2, p_3, for the coefficients of $\mathfrak{R}_1$, $\mathfrak{R}_2$, $\mathfrak{R}_3$ in I, and $\mathfrak{T}$ for the error of the equation, we have exactly

$$p_1\mathfrak{R}_1-p_2\mathfrak{R}_2+p_3\mathfrak{R}_3=\mathfrak{T},$$

which gives

$$p_1\mathfrak{R}_1\times\mathfrak{R}_3-p_2\mathfrak{R}_2\times\mathfrak{R}_3=\mathfrak{T}\times\mathfrak{R}_3,$$

$$\frac{\mathfrak{R}_2\times\mathfrak{R}_3}{\mathfrak{R}_1\times\mathfrak{R}_3}=\frac{p_1}{p_2}-\frac{\mathfrak{T}\times\mathfrak{R}_3}{p_2\mathfrak{R}_1\times\mathfrak{R}_3}.$$

Now $\frac{p_1}{p_2}$ is my expression for the ratio of the triangles, and $\frac{\mathfrak{T}\times\mathfrak{R}_3}{p_2\mathfrak{R}_1\times\mathfrak{R}_3}$ is its error. This is of the fourth order in general (since the denominator is of the first), and for equal intervals, of the fifth. The same is true of the two other ratios. Thus we have

$$\frac{\mathfrak{R}_1\times\mathfrak{R}_2}{\mathfrak{R}_1\times\mathfrak{R}_3}=\frac{p_3}{p_2}-\frac{\mathfrak{R}_1\times\mathfrak{T}}{p_2\mathfrak{R}_1\times\mathfrak{R}_3}.$$

Adding these equations and subtracting 1 [from both sides] we have

$$\frac{\mathfrak{R}_1\times\mathfrak{R}_2+\mathfrak{R}_2\times\mathfrak{R}_3-\mathfrak{R}_1\times\mathfrak{R}_3}{\mathfrak{R}_1\times\mathfrak{R}_3}=\frac{p_1+p_3-p_2}{p_2}+\frac{(\mathfrak{R}_3-\mathfrak{R}_1)\times\mathfrak{T}}{p_2\mathfrak{R}_1\times\mathfrak{R}_3}.$$

Here the last term, which represents the error, is of the fifth order in general, or for equal intervals, of the sixth. But the

quantity sought is of the second order, and the *relative* error is of the third order in the general case, or the fourth for equal intervals. It is precisely this error which is most important in the case of elliptic orbits.

It will be observed that the accuracy of the expressions for the ratios $[r_1r_2]:[r_2r_3]:[r_1r_3]$ affords no measure of the accuracy of the formula for the determination of elliptic orbits.

I think that this hasty sketch will illustrate the convenience and perspicuity of vector notations in this subject, quite independently of any particular method which is chosen for the determination of the orbit. What is the best method? is hardly, I think, a question which admits of a definite reply. It certainly depends upon the ratio of the time intervals, their absolute value, and many other things.

Yours very truly,

J. WILLARD GIBBS.

P.S.—If we wish to use the *curtate* distances, with reference to the ecliptic or the equator, let ρ_1 be defined as the distance multiplied by cosine (lat. or dec.), and $\mathfrak{F}_1$ as a vector of length secant (lat. or dec.). For the most part the formulæ will require no change, but the square of $\mathfrak{F}_1$ will be $\sec^2$(lat. or dec.) instead of unity, so that the last terms of (8) will have this factor. $(\mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3)$ will then be Gauss' (0.1.2.), whereas in my paper $(\mathfrak{F}_1\mathfrak{F}_2\mathfrak{F}_3)$ is Lagrange's $(C'C''C''')$.

J. W. G.

VII.

ON THE RÔLE OF QUATERNIONS IN THE ALGEBRA OF VECTORS.

[*Nature*, vol. XLIII. pp. 511–513, April 2, 1891.]

THE following passage, which has recently come to my notice, in the preface to the third edition of Prof. Tait's *Quaternions* seems to call for some reply:

"Even Prof. Willard Gibbs must be ranked as one of the retarders of quaternion progress, in virtue of his pamphlet on *Vector Analysis*, a sort of hermaphrodite monster, compounded of the notations of Hamilton and of Grassmann."

The merits or demerits of a pamphlet printed for private distribution a good many years ago do not constitute a subject of any great importance, but the assumptions implied in the sentence quoted are suggestive of certain reflections and inquiries which are of broader interest, and seem not untimely at a period when the methods and results of the various forms of multiple algebra are attracting so much attention. It seems to be assumed that a departure from quaternionic usage in the treatment of vectors is an enormity. If this assumption is true, it is an important truth; if not, it would be unfortunate if it should remain unchallenged, especially when supported by so high an authority. The criticism relates particularly to notations, but I believe that there is a deeper question of notions underlying that of notations. Indeed, if my offence had been solely in the matter of notation, it would have been less accurate to describe my production as a monstrosity, than to characterize its dress as uncouth.

Now what are the fundamental notions which are germane to a vector analysis? (A vector analysis is of course an algebra for vectors, or something which shall be to vectors what ordinary algebra is to ordinary quantities.) If we pass over those notions which are so simple that they go without saying, geometrical addition (denoted by $+$) is, perhaps, first to be mentioned. Then comes the product of the lengths of two vectors and the cosine of the angle which they include. This, taken negatively, is denoted in quaternions by $S\alpha\beta$, where α and β are the vectors. Equally important is a vector at right angles to α and β (on a specified side of their plane), and

representing in length the product of their lengths and the sine of the angle which they include. This is denoted by $V\alpha\beta$ in quaternions. How these notions are represented in my pamphlet is a question of very subordinate consequence, which need not be considered at present. The importance of these notions, and the importance of a suitable notation for them, is not, I suppose, a matter on which there is any difference of opinion. Another function of α and β, called their product and written $\alpha\beta$, is used in quaternions. In the general case, this is neither a vector, like $V\alpha\beta$, nor a scalar (or ordinary algebraic quantity), like $S\alpha\beta$, but a quaternion—that is, it is part vector and part scalar. It may be defined by the equation—

$$\alpha\beta = V\alpha\beta + S\alpha\beta.$$

The question arises, whether the quaternionic product can claim a prominent and fundamental place in a system of vector analysis. It certainly does not hold any such place among the fundamental geometrical conceptions as the geometrical sum, the scalar product, or the vector product. The geometrical sum $\alpha+\beta$ represents the third side of a triangle as determined by the sides α and β. $V\alpha\beta$ represents in magnitude the area of the parallelogram determined by the sides α and β, and in direction the normal to the plane of the parallelogram. $S\gamma V\alpha\beta$ represents the volume of the parallelopiped determined by the edges α, β, and γ. These conceptions are the very foundations of geometry.

We may arrive at the same conclusion from a somewhat narrower but very practical point of view. It will hardly be denied that sines and cosines play the leading parts in trigonometry. Now the notations $V\alpha\beta$ and $S\alpha\beta$ represent the sine and the cosine of the angle included between α and β, combined in each case with certain other simple notions. But the sine and cosine combined with these auxiliary notions are incomparably more amenable to analytical transformation than the simple sine and cosine of trigonometry, exactly as numerical quantities combined (as in algebra) with the notion of positive or negative quality are incomparably more amenable to analytical transformation than the simple numerical quantities of arithmetic.

I do not know of anything which can be urged in favor of the quaternionic product of two vectors as a *fundamental* notion in vector analysis, which does not appear trivial or artificial in comparison with the above considerations. The same is true of the quaternionic quotient, and of the quaternion in general.

How much more deeply rooted in the nature of things are the functions $S\alpha\beta$ and $V\alpha\beta$ than any which depend on the definition of a quaternion, will appear in a strong light if we try to extend

our formulæ to space of four or more dimensions. It will not be claimed that the notions of quaternions will apply to such a space, except indeed in such a limited and artificial manner as to rob them of their value as a system of geometrical algebra. But vectors exist in such a space, and there must be a vector analysis for such a space. The notions of geometrical addition and the scalar product are evidently applicable to such a space. As we cannot define the direction of a vector in space of four or more dimensions by the condition of perpendicularity to two given vectors, the definition of $V\alpha\beta$, as given above, will not apply *totidem verbis* to space of four or more dimensions. But a little change in the definition, which would make no essential difference in three dimensions, would enable us to apply the idea at once to space of any number of dimensions.

These considerations are of a somewhat *a priori* nature. It may be more convincing to consider the use actually made of the quaternion as an instrument for the expression of spatial relations. The principal use seems to be the derivation of the functions expressed by $S\alpha\beta$ and $V\alpha\beta$. Each of these expressions is regarded by quaternionic writers as representing two distinct operations; first, the formation of the product $\alpha\beta$, which is the quaternion, and then the taking out of this quaternion the scalar or the vector part, as the case may be, this second process being represented by the selective symbol, S or V. This is, I suppose, the natural development of the subject in a treatise on quaternions, where the chosen subject seems to require that we should commence with the idea of a quaternion, or get there as soon as possible, and then develop everything from that particular point of view. In a system of vector analysis, in which the principle of development is not thus predetermined, it seems to me contrary to good method that the more simple and elementary notions should be defined by means of those which are less so.

The quaternion affords a convenient notation for rotations. The notation $q(\ )q^{-1}$, where q is a quaternion and the operand is to be written in the parenthesis, produces on all possible vectors just such changes as a (finite) rotation of a solid body. Rotations may also be represented, in a manner which seems to leave nothing to be desired, by linear vector functions. Doubtless each method has advantages in certain cases, or for certain purposes. But since nothing is more simple than the definition of a linear vector function, while the definition of a quaternion is far from simple, and since in any case linear vector functions must be treated in a system of vector analysis, capacity for representing rotations does not seem to me sufficient to entitle the quaternion to a place among the *fundamental* and *necessary* notions of a vector analysis.

Another use of the quaternionic idea is associated with the symbol ∇.

The quantities written $S\nabla\omega$ and $V\nabla\omega$, where ω denotes a vector having values which vary in space, are of fundamental importance in physics. In quaternions these are derived from the quaternion $\nabla\omega$ by selecting respectively the scalar or the vector part. But the most simple and elementary definitions of $S\nabla\omega$ and $V\nabla\omega$ are quite independent of the conception of a quaternion, and the quaternion $\nabla\omega$ is scarcely used except in combination with the symbols S and V, expressed or implied. There are a few formulæ in which there is a trifling gain in compactness in the use of the quaternion, but the gain is very trifling so far as I have observed, and generally, it seems to me, at the expense of perspicuity.

These considerations are sufficient, I think, to show that the position of the quaternionist is not the only one from which the subject of vector analysis may be viewed, and that a method which would be monstrous from one point of view, may be normal and inevitable from another.

Let us now pass to the subject of notations. I do not know wherein the notations of my pamphlet have any special resemblance to Grassmann's, although the point of view from which the pamphlet was written is certainly much nearer to his than to Hamilton's. But this a matter of minor consequence. It is more important to ask, What are the requisites of a good notation for the purposes of vector analysis? There is no difference of opinion about the representation of geometrical addition. When we come to functions having an analogy to multiplication, the products of the lengths of two vectors and the cosine of the angle which they include, from any point of view except that of the quaternionist, seems more simple than the same quantity taken negatively. Therefore we want a notation for what is expressed by $-S\alpha\beta$, rather than $S\alpha\beta$, in quaternions. Shall the symbol denoting this function be a letter or some other sign? and shall it precede the vectors or be placed between them? A little reflection will show, I think, that while we must often have recourse to letters to supplement the number of signs available for the expression of all kinds of operations, it is better that the symbols expressing the most fundamental and frequently recurring operations should not be letters, and that a sign between the vectors, and, as it were, uniting them, is better than a sign before them in a case having a formal analogy with multiplication. The case may be compared with that of addition, for which $\alpha+\beta$ is evidently more convenient than $\Sigma(\alpha, \beta)$ or $\Sigma\alpha\beta$ would be. Similar considerations will apply to the function written in quaternions $V\alpha\beta$. It would seem that we obtain the *ne plus ultra* of simplicity and convenience, if we express the two functions by uniting the vectors in each case with a sign suggestive of multiplication. The particular forms of the signs which

we adopt is a matter of minor consequence. In order to keep within the resources of an ordinary printing office, I have used a dot and a cross, which are already associated with multiplication, but are not needed for ordinary multiplication, which is best denoted by the simple juxtaposition of the factors. I have no especial predilection for these particular signs. The use of the dot is indeed liable to the objection that it interferes with its use as a separatrix, or instead of a parenthesis.

If, then, I have written $\alpha.\beta$ and $\alpha\times\beta$ for what is expressed in quaternions by $-\mathrm{S}\alpha\beta$ and $\mathrm{V}\alpha\beta$, and in like manner $\nabla.\omega$ and $\nabla\times\omega$ for $-\mathrm{S}\nabla\omega$ and $\mathrm{V}\nabla\omega$ in quaternions, it is because the natural development of a vector analysis seemed to lead logically to some such notations. But I think that I can show that these notations have some substantial advantages over the quaternionic in point of convenience.

Any linear vector function of a variable vector ρ may be expressed in the form—

$$\alpha\lambda.\rho+\beta\mu.\rho+\gamma\nu.\rho=(\alpha\lambda+\beta\mu+\gamma\nu).\rho=\Phi.\rho,$$

where $$\Phi=\alpha\lambda+\beta\mu+\gamma\nu;$$

or in quaternions

$$-\alpha\mathrm{S}\lambda\rho-\beta\mathrm{S}\mu\rho-\gamma\mathrm{S}\nu\rho=-(\alpha\mathrm{S}\lambda+\beta\mathrm{S}\mu+\gamma\mathrm{S}\nu)\rho=-\phi\rho,$$

where $$\phi=\alpha\mathrm{S}\lambda+\beta\mathrm{S}\mu+\gamma\mathrm{S}\nu.$$

If we take the scalar product of the vector $\Phi.\rho$, and another vector σ, we obtain the scalar quantity

$$\sigma.\Phi.\rho=\sigma.(\alpha\lambda+\beta\mu+\gamma\nu).\rho,$$

or in quaternions $$\mathrm{S}\sigma\phi\rho=\mathrm{S}\sigma(\alpha\mathrm{S}\lambda+\beta\mathrm{S}\mu+\gamma\mathrm{S}\nu)\rho.$$

This is a function of σ and of ρ, and it is exactly the same kind of function of σ that it is of ρ, a symmetry which is not so clearly exhibited in the quaternionic notation as in the other. Moreover, we can write $\sigma.\Phi$ for $\sigma.(\alpha\lambda+\beta\mu+\gamma\nu)$. This represents a vector which is a function of σ, viz., the function conjugate to $\Phi.\sigma$; and $\sigma.\Phi.\rho$ may be regarded as the product of this vector and ρ. This is not so clearly indicated in the quaternionic notation, where it would be straining things a little to call $\mathrm{S}\sigma\phi$ a vector.

The combinations $\alpha\lambda$, $\beta\mu$, etc., used above, are distributive with regard to each of the two vectors, and may be regarded as a kind of product. If we wish to express everything in terms of i, j, and k, Φ will appear as a sum of ii, ij, ik, ji, jj, jk, ki, kj, kk, each with a numerical coefficient. These nine coefficients may be arranged in a square, and constitute a matrix; and the study of the properties of expressions like Φ is identical with the study of ternary matrices. This expression of the matrix as a sum of products (which may be

extended to matrices of any order) affords a point of departure from which the properties of matrices may be deduced with the utmost facility. The ordinary matricular product is expressed by a dot, as $\Phi.\Psi$. Other important kinds of multiplication may be defined by the equations—

$$(a\lambda)^{\times}_{\times}(\beta\mu)=(a\times\beta)(\lambda\times\mu), \qquad (a\lambda):(\beta\mu)=(a.\beta)(\lambda.\mu).$$

With these definitions $\frac{1}{6}\Phi^{\times}_{\times}\Phi:\Phi$ will be the determinant of Φ, and $\Phi^{\times}_{\times}\Phi$ will be the conjugate of the reciprocal of Φ multiplied by twice the determinant. If Φ represents the manner in which vectors are affected by a strain, $\frac{1}{2}\Phi^{\times}_{\times}\Phi$ will represent the manner in which surfaces are affected, and $\frac{1}{6}\Phi^{\times}_{\times}\Phi:\Phi$ the manner in which volumes are affected. Considerations of this kind do not attach themselves so naturally to the notation $\phi=a\mathrm{S}\lambda+\beta\mathrm{S}\mu+\gamma\mathrm{S}\nu$, nor does the subject admit so free a development with this notation, principally because the symbol S refers to a special use of the matrix, and is very much in the way when we want to apply the matrix to other uses, or to subject it to various operations.

VIII.

QUATERNIONS AND THE *AUSDEHNUNGSLEHRE.*

[*Nature*, vol. XLIV. pp. 79-82, May 28, 1891.]

THE year 1844 is memorable in the annals of mathematics on account of the first appearance on the printed page of Hamilton's *Quaternions* and Grassmann's *Ausdehnungslehre*. The former appeared in the July, October, and supplementary numbers of the *Philosophical Magazine*, after a previous communication to the Royal Irish Academy, November 13, 1843. This communication was indeed announced to the Council of the Academy four weeks earlier, on the very day of Hamilton's discovery of quaternions, as we learn from one of his letters. The author of the *Ausdehnungslehre*, although not unconscious of the value of his ideas, seems to have been in no haste to place himself on record, and published nothing until he was able to give the world the most characteristic and fundamental part of his system with considerable development in a treatise of more than 300 pages, which appeared in August 1844.

The doctrine of quaternions has won a conspicuous place among the various branches of mathematics, but the nature and scope of the *Ausdehnungslehre*, and its relation to quaternions, seem to be still the subject of serious misapprehension in quarters where we naturally look for accurate information. Historical justice, and the interests of mathematical science, seem to require that the allusions to the *Ausdehnungslehre* in the article on "Quaternions" in the last edition of the *Encyclopædia Britannica*, and in the third edition of Prof. Tait's *Treatise on Quaternions*, should not be allowed to pass without protest.

It is principally as systems of geometrical algebra that quaternions and the *Ausdehnungslehre* come into comparison. To appreciate the relations of the two systems, I do not see how we can proceed better than if we ask first what they have in common, then what either system possesses which is peculiar to itself. The relative extent and importance of the three fields, that which is common to the two systems, and those which are peculiar to each, will determine the relative rank of the geometrical algebras. Questions of priority can only relate to the field common to both, and will be much simplified by having the limits of that field clearly drawn.

Geometrical addition in three dimensions is common to the two systems, and seems to have been discovered independently both by Hamilton and Grassmann, as well as by several other persons about the same time. It is not probable that any especial claim for priority with respect to this principle will be urged for either of the two with which we are now concerned.

The functions of two vectors which are represented in quaternions by $S\alpha\beta$ and $V\alpha\beta$ are common to both systems as published in 1844, but the quaternion is peculiar to Hamilton's. The linear vector function is common to both systems as ultimately developed, although mentioned only by Grassmann as early as 1844.

To those already acquainted with quaternions, the first question will naturally be: To what extent are the geometrical methods which are usually called quaternionic peculiar to Hamilton, and to what extent are they common to Grassmann? This is a question which anyone can easily decide for himself. It is only necessary to run one's eye over the equations used by quaternionic writers in the discussion of geometrical or physical subjects, and see how far they necessarily involve the idea of the quaternion, and how far they would be intelligible to one understanding the functions $S\alpha\beta$ and $V\alpha\beta$, but having no conception of the quaternion $\alpha\beta$, or at least could be made so by trifling changes of notation, as by writing S or V in places where they would not affect the value of the expressions. For such a test the examples and illustrations in treatises on quaternions would be manifestly inappropriate, so far as they are chosen to illustrate quaternionic principles, since the object may influence the form of presentation. But we may use any discussion of geometrical or physical subjects, where the writer is free to choose the form most suitable to the subject. I myself have used the chapters and sections in Prof. Tait's *Quaternions* on the following subjects: Geometry of the straight line and plane, the sphere and cyclic cone, surfaces of the second degree, geometry of curves and surfaces, kinematics, statics and kinetics of a rigid system, special kinetic problems, geometrical and physical optics, electrodynamics, general expressions for the action between linear elements, application of ∇ to certain physical analogies, pp. 160-371, except the examples (not worked out) at the close of the chapters.

Such an examination will show that for the most part the methods of representing spatial relations used by quaternionic writers are common to the systems of Hamilton and Grassmann. To an extent comparatively limited, cases will be found in which the quaternionic idea forms an essential element in the signification of the equations.

The question will then arise with respect to the comparatively limited field which is the peculiar property of Hamilton, How im-

portant are the advantages to be gained by the use of the quaternion? This question, unlike the preceding, is one into which a personal equation will necessarily enter. Everyone will naturally prefer the methods with which he is most familiar; but I think that it may be safely affirmed that in the majority of cases in this field the advantage derived from the use of the quaternion is either doubtful or very trifling. There remains a residuum of cases in which a substantial advantage is gained by the use of the quaternionic method. Such cases, however, so far as my own observation and experience extend, are very exceptional. If a more extended and careful inquiry should show that they are ten times as numerous as I have found them, they would still be exceptional.

We have now to inquire what we find in the *Ausdehnungslehre* in the way of a geometrical algebra, that is wanting in quaternions. In addition to an algebra of vectors, the *Ausdehnungslehre* affords a system of geometrical algebra in which the point is the fundamental element, and which for convenience I shall call Grassmann's algebra of points. In this algebra we have first the addition of points, or quantities located at points, which may be explained as follows. The equation

$$aA+bB+cC+\text{ etc.}=eE+fF+\text{ etc.},$$

in which the capitals denote points, and the small letters scalars (or ordinary algebraic quantities), signifies that

$$a+b+c+\text{ etc.}=e+f+\text{ etc.},$$

and also that the centre of gravity of the weights a, b, c, etc., at the points A, B, C, etc., is the same as that of the weights e, f, etc., at the points E, F, etc. (It will be understood that negative weights are allowed as well as positive.) The equation is thus equivalent to four equations of ordinary algebra. In this Grassmann was anticipated by Möbius (*Barycentrischer Calcul*, 1827).

We have next the addition of finite straight lines, or quantities located in straight lines (*Liniengrössen*). The meaning of the equation

$$AB+CD+\text{ etc.}=EF+GH+\text{ etc.}$$

will perhaps be understood most readily, if we suppose that each member represents a system of forces acting on a rigid body. The equation then signifies that the two systems are equivalent. An equation of this form is therefore equivalent to six ordinary equations. It will be observed that the *Liniengrössen* AB and CD are not simply vectors; they have not merely length and direction, but they are also located each in a given line, although their position within those lines is immaterial. In Clifford's terminology, AB is a *rotor*, AB+CD a *motor*. In the language of Prof. Ball's *Theory of Screws*, AB+CD represents either a *twist* or a *wrench*.

We have next the addition of plane surfaces (*Plangrössen*). The equation

$$ABC+DEF+GHI=JKL$$

signifies that the plane JKL passes through the point common to the planes ABC, DEF, and GHI, and that the projection by parallel lines of the triangle JKL on any plane is equal to the sum of the projections of ABC, DEF, and GHI on the same plane, the areas being taken positively or negatively according to the cyclic order of the projected points. This makes the equation equivalent to four ordinary equations.

Finally, we have the addition of volumes, as in the equation

$$ABCD+EFGH=IJKL,$$

where there is nothing peculiar, except that each term represents the six-fold volume of the tetrahedron, and is to be taken positively or negatively according to the relative position of the points.

We have also multiplications as follows: The line (*Liniengrösse*) AB is regarded as the product of the points A and B. The *Plangrösse* ABC, which represents the double area of the triangle, is regarded as the product of the three points A, B, and C, or as the product of the line AB and the point C, or of BC and A, or indeed of BA and C. The volume ABCD, which represents six times the tetrahedron, is regarded as the product of the points A, B, C, and D, or as the product of the point A and the *Plangrösse* BCD, or as the product of the lines AB and BC, etc., etc.

This does not exhaust the wealth of multiplicative relations which Grassmann has found in the very elements of geometry. The following products are called *regressive*, as distinguished from the *progressive*, which have been described. The product of the *Plangrössen* ABC and DEF is a part of the line in which the planes ABC and DEF intersect, which is equal in numerical value to the product of the double areas of the triangles ABC and DEF multiplied by the sine of the angle made by the planes. The product of the *Liniengrösse* AB and the *Plangrösse* CDE is the point of intersection of the line and the plane with a numerical coefficient representing the product of the length of the line and the double area of the triangle multiplied by the sine of the angle made by the line and the plane The product of three *Plangrössen* is consequently the point common to the three planes with a certain numerical coefficient. In plane geometry we have a regressive product of two *Liniengrössen*, which gives the point of intersection of the lines with a certain numerical coefficient.

The fundamental operations relating to the point, line, and plane are thus translated into analysis by multiplications. The immense flexibility and power of such an analysis will be appreciated by

anyone who considers what generalized multiplication in connection with additive relations has done in other fields, as in quaternions, or in the theory of matrices, or in the algebra of logic. For a single example, if we multiply the equation

$$AB+CD+\text{etc.}=EF+GH+\text{etc.}$$

by PQ (P and Q being any two points), we have

$$ABPQ+CDPQ+\text{etc.}=EFPQ+GHPQ+\text{etc.},$$

which will be recognised as expressing an important theorem of statics.

The field in which Grassmann's algebra of points, as distinguished from his algebra of vectors, finds its especial application and utility is nearly coincident with that in which, when we use the methods of ordinary algebra, tetrahedral or anharmonic coordinates are more appropriate than rectilinear. In fact, Grassmann's algebra of points may be regarded as the application of the methods of multiple algebra to the notions connected with tetrahedral coordinates, just as his or Hamilton's algebra of vectors may be regarded as the application of the methods of multiple algebra to the notions connected with rectilinear coordinates. These methods, however, enrich the field to which they are applied with new notions. Thus the notion of the coordinates of a line in space, subsequently introduced by Plücker, was first given in the *Ausdehnungslehre* of 1844. It should also be observed that the utility of a multiple algebra when it takes the place of an ordinary algebra of four coordinates, is very much greater than when it takes the place of three coordinates, for the same reason that a multiple algebra taking the place of three coordinates is very much more useful than one taking the place of two. Grassmann's algebra of points will always command the admiration of geometers and analysts, and furnishes an instrument of marvellous power to the former, and in its general form, as applicable to space of any number of dimensions, to the latter. To the physicist an algebra of points is by no means so indispensable an instrument as an algebra of vectors.

Grassmann's algebra of vectors, which we have described as coincident with a part of Hamilton's system, is not really anything separate from his algebra of points, but constitutes a part of it, the vector arising when one point is subtracted from another. Yet it constitutes a whole, complete in itself, and we may separate it from the larger system to facilitate comparison with the methods of Hamilton.

We have, then, as geometrical algebras published in 1844, an algebra of vectors common to Hamilton and Grassmann, augmented on Hamilton's side by the quaternion, and on Grassmann's by his algebra of points. This statement should be made with the

reservation that the *addition* both of vectors and of points had been given by earlier writers.

In both systems as finally developed we have the linear vector function, the theory of which is identical with that of strains and rotations. In Hamilton's system we have also the linear quaternion function, and in Grassmann's the linear function applied to the quantities of his algebra of points. This application gives those transformations in which projective properties are preserved, the doctrine of reciprocal figures or principle of duality, etc. (Grassmann's theory of the linear function is, indeed, broader than this, being coextensive with the theory of matrices; but we are here considering only the geometrical side of the theory.)

In his earliest writings on quaternions, Hamilton does not discuss the linear function. In his *Lectures on Quaternions* (1853), he treats of the inversion of the linear vector function, as also of the linear quaternion function, and shows how to find the latent roots of the vector function, with the corresponding axes for the case of real and unequal roots. He also gives a remarkable equation, *the symbolic cubic*, which the functional symbol must satisfy. This equation is a particular case of that which is given in Prof. Cayley's classical *Memoir on the Theory of Matrices* (1858), and which is called by Prof. Sylvester the Hamilton-Cayley equation. In his *Elements of Quaternions* (1866), Hamilton extends the symbolic equation to the quaternion function.

In Grassmann, although the linear function is mentioned in the first *Ausdehnungslehre*, we do not find so full a discussion of the subject until the second *Ausdehnungslehre* (1862), where he discusses the latent roots and axes, or what corresponds to axes in the general theory, the whole discussion relating to matrices of any order. The more difficult cases are included, as that of a strain in which all the roots are real, but there is only one axis or unchanged direction. On the formal side he shows how a linear function may be represented by a quotient or sum of quotients, and by a sum of products, *Lückenausdruck*.

More important, perhaps, than the question when this or that theorem was first published is the question where we first find those notions and notations which give the key to the algebra of linear functions, or the algebra of matrices, as it is now generally called. In vol. xxxi, p. 35, of *Nature*, Prof. Sylvester speaks of Cayley's "ever-memorable" *Memoir on Matrices* as constituting "a second birth of Algebra, its *avatar* in a new and glorified form," and refers to a passage in his *Lectures on Universal Algebra*, from which, I think, we are justified in inferring that this characterization of the memoir is largely due to the fact that it is there shown how matrices

may be treated as extensive quantities, capable of addition as well as of multiplication. This idea, however, is older than the memoir of 1858. The *Lückenausdruck*, by which the matrix is expressed as a sum of a kind of products (*lückenhaltig*, or open), is described in a note at the end of the first *Ausdehnungslehre*. There we have the matrix given not only as a sum, but as a sum of products, introducing a multiplicative relation entirely different from the ordinary multiplication of matrices, and hardly less fruitful, but not lying nearly so near the surface as the relations to which Prof. Sylvester refers. The key to the theory of matrices is certainly given in the first *Ausdehnungslehre*, and if we call the birth of matricular analysis the second birth of algebra, we can give no later date to this event than the memorable year of 1844.

The immediate occasion of this communication is the following passage in the preface to the third edition of Prof. Tait's *Quaternions*:

"Hamilton not only published his theory complete, the year before the first (and extremely imperfect) sketch of the *Ausdehnungslehre* appeared; but had given ten years before, in his protracted study of Sets, the very processes of external and internal multiplication (corresponding to the Vector and Scalar parts of a product of two vectors) which have been put forward as specially the property of Grassmann."

For additional information we are referred to art. "Quaternions," *Encyc. Brit.*, where we read respecting the first *Ausdehnungslehre*:

"In particular two species of multiplication ('inner' and 'outer') of directed lines in one plane were given. The results of these two kinds of multiplication correspond respectively to the numerical and the directed parts of Hamilton's quaternion product. But Grassmann distinctly states in his preface that he had not had leisure to extend his method to angles in space. . . . But his claims, however great they may be, can in no way conflict with those of Hamilton, whose mode of multiplying *couples* (in which the 'inner' and 'outer' multiplication are essentially involved) was produced in 1833, and whose quaternion system was completed and published before Grassmann had elaborated for press even the rudimentary portions of his own system, in which the veritable difficulty of the whole subject, the application to angles in space, had not even been attacked."

I shall leave the reader to judge of the accuracy of the general terms used in these passages in comparing the first *Ausdehnungslehre* with Hamilton's system as published in 1843 or 1844. The specific statements respecting Hamilton and Grassmann require an answer.

It must be Hamilton's *Theory of Conjugate Functions or Algebraic Couples* (read to the Royal Irish Academy, 1833 and 1835, and published in vol. xvii of the *Transactions*) to which reference is made in the statements concerning his "protracted study of Sets" and

"mode of multiplying *couples.*" But I cannot find anything like Grassmann's external or internal multiplication in this memoir, which is concerned, as the title pretty clearly indicates, with the theory of the complex quantities of ordinary algebra.

It is difficult to understand the statements respecting the *Ausdehnungslehre,* which seem to imply that Grassmann's two kinds of multiplication were subject to some kind of limitation to a plane. The external product is not limited in the first *Ausdehnungslehre* even to three dimensions. The internal, which is a comparatively simple matter, is mentioned in the first *Ausdehnungslehre* only in the preface, where it is defined, and placed beside the external product as relating to directed lines. There is not the least suggestion of any difference in the products in respect to the generality of their application to vectors.

The misunderstanding seems to have arisen from the following sentence in Grassmann's preface: "And in general, in the consideration of angles in space, difficulties present themselves, for the complete (*allseitig*) solution of which I have not yet had sufficient leisure." It is not surprising that Grassmann should have required more time for the development of some parts of his system, when we consider that Hamilton, on his discovery of quaternions, estimated the time which he should wish to devote to them at ten or fifteen years (see his letter to Prof. Tait in the *North British Review* for September 1866), and actually took several years to prepare for the press as many pages as Grassmann had printed in 1844. But any speculation as to the questions which Grassmann may have had principally in mind in the sentence quoted, and the particular nature of the difficulties which he found in them, however interesting from other points of view, seems a very precarious foundation for a comparison of the systems of Hamilton and Grassmann as published in the years 1843-44. Such a comparison should be based on the positive evidence of doctrines and methods actually published.

Such a comparison I have endeavoured to make, or rather to indicate the basis on which it may be made, so far as systems of geometrical algebra are concerned. As a contribution to analysis in general, I suppose that there is no question that Grassmann's system is of indefinitely greater extension, having no limitation to any particular number of dimensions.

IX.

QUATERNIONS AND THE ALGEBRA OF VECTORS.

[*Nature*, vol. XLVII. pp. 463, 464, Mar. 16, 1893.]

In a recent number of *Nature* [vol. xlvii, p. 151], Mr. McAulay puts certain questions to Mr. Heaviside and to me, relating to a subject of such importance as to justify an answer somewhat at length. I cannot of course speak for Mr. Heaviside, although I suppose that his views are not very different from mine on the most essential points, but even if he shall have already replied before this letter can appear, I shall be glad to add whatever of force may belong to independent testimony.

Mr. McAulay asks: "What is the *first* duty of the physical vector analyst *quâ* physical vector analyst?" The answer is not doubtful. It is to present the subject in such a form as to be most easily acquired, and most useful when acquired.

In regard to the slow progress of such methods towards recognition and use by physicists and others, which Mr. McAulay deplores, it does not seem possible to impute it to any want of uniformity of notation. I doubt whether there is any modern branch of mathematics which has been presented for so long a time with a greater uniformity of notation than quaternions.

What, then, is the cause of the fact which Mr. McAulay and all of us deplore? It is not far to seek. We need only a glance at the volumes in which Hamilton set forth his method. No wonder that physicists and others failed to perceive the possibilities of simplicity, perspicuity, and brevity which were contained in a system presented to them in ponderous volumes of 800 pages. Perhaps Hamilton may have intended these volumes as a sort of *thesaurus*, and we should look to his shorter papers for a compact account of his method. But if we turn to his earlier papers on Quaternions in the *Philosophical Magazine*, in which principally he introduced the subject to the notice of his contemporaries, we find them entitled "On Quaternions; or on a New System of Imaginaries in Algebra," and in them we find a great deal about imaginaries, and very little of a vector analysis. To show how slowly the system of vector analysis developed itself in the quaternionic *nidus*, we need only say that the symbols S, V, and ∇ do not appear until two or three years after the discovery of quaternions. In short, it seems to have been

only a secondary object with Hamilton to express the geometrical relations of vectors,—secondary in time, and also secondary in this, that it was never allowed to give shape to his work.

But this relates to the past. In regard to the present *status*, I beg leave to quote what Mr. McAulay has said on another occasion (see *Phil. Mag.*, June 1892):—"Quaternions differ in an important respect from other branches of mathematics that are studied by mathematicians after they have in the course of years of hard labour laid the foundation of all their future work. In nearly all cases these branches are very properly so called. They each grow out of a definite spot of the main tree of mathematics, and derive their sustenance from the sap of the trunk as a whole. But not so with quaternions. To let these grow in the brain of a mathematician, he must start from the seed as with the rest of his mathematics regarded as a whole. He cannot graft them on his already flourishing tree, for they will die there. They are independent plants that require separate sowing and the consequent careful tending."

Can we wonder that mathematicians, physicists, astronomers, and geometers feel some doubt as to the value or necessity of something so separate from all other branches of learning? Can that be a natural treatment of the subject which has no relations to any other method, and, as one might suppose from reading some treatises, has only occurred to a single man? Or, at best, is it not discouraging to be told that in order to use the quaternionic method, one must give up the progress which he has already made in the pursuit of his favourite science, and go back to the beginning and start anew on a parallel course?

I believe, however, that if what I have quoted is true of vector methods, it is because there is something fundamentally wrong in the presentation of the subject. Of course, in some sense and to some extent it is and must be true. Whatever is special, accidental, and individual, will die, as it should; but that which is universal and essential should remain as an organic part of the whole intellectual acquisition. If that which is essential dies with the accidental, it must be because the accidental has been given the prominence which belongs to the essential. For myself, I should preach no such doctrine to those whom I wish to convert to the true faith.

In Italy, they say, all roads lead to Rome. In mechanics, kinematics, astronomy, physics, all study leads to the consideration of certain relations and operations. These are the capital notions; these should have the leading parts in any analysis suited to the subject.

If I wished to attract the student of any of these sciences to an algebra for vectors, I should tell him that the fundamental notions of this algebra were exactly those with which he was daily con-

versant. I should tell him that a vector algebra is so far from being any one man's production that half a century ago several were already working toward an algebra which should be primarily geometrical and not arithmetical, and that there is a remarkable similarity in the results to which these efforts led (see *Proc. A.A.A.S.* for 1886, pp. 37, ff.) [this vol. p. 91, ff.]. I should call his attention to the fact that Lagrange and Gauss used the notation $(\alpha\beta\gamma)$ to denote precisely the same as Hamilton by his $S(\alpha\beta\gamma)$, except that Lagrange limited the expression to unit vectors, and Gauss to vectors of which the length is the secant of the latitude, and I should show him that we have only to give up these limitations, and the expression (in connection with the notion of geometrical addition) is endowed with an immense wealth of transformations. I should call his attention to the fact that the notation $[r_1 r_2]$, universal in the theory of orbits, is identical with Hamilton's $V(\rho_1\rho_2)$, except that Hamilton takes the area as a vector, i.e., includes the notion of the direction of the normal to the plane of the triangle, and that with this simple modification (and with the notion of geometrical addition of surfaces as well as of lines) this expression becomes closely connected with the first-mentioned, and is not only endowed with a similar capability for transformation, but enriches the first with new capabilities. In fact, I should tell him that the notions which we use in vector analysis are those which he who reads between the lines will meet on every page of the great masters of analysis, or of those who have probed deepest the secrets of nature, the only difference being that the vector analyst, having regard to the weakness of the human intellect, does as the early painters who wrote beneath their pictures "This is a tree," "This is a horse."

I cannot attach quite so much importance as Mr. McAulay to uniformity of notation. That very uniformity, if it existed among those who use a vector analysis, would rather obscure than reveal their connection with the general course of modern thought in mathematics and physics. There are two ways in which we may measure the progress of any reform. The one consists in counting those who have adopted the *shibboleth* of the reformers; the other measure is the degree in which the community is imbued with the essential principles of the reform. I should apply the broader measure to the present case, and do not find it quite so bad as Mr. McAulay does.

Yet the question of notations, although not the vital question, is certainly important, and I assure Mr. McAulay that reluctance to make unnecessary innovations in notation has been a very powerful motive in restraining me from publication. Indeed my pamphlet on *Vector Analysis*, which has excited the animadversion of quaternionists, was never formally published, although rather widely

distributed, so long as I had copies to distribute, among those who I thought might be interested in the subject. I may say, however, since I am called upon to defend my position, that I have found the notations of that pamphlet more flexible than those generally used. Mr. McAulay, at least, will understand what I mean by this, if I say that some of the relations which he has thought of sufficient importance to express by means of special devices (see *Proc. R.S.E.* for 1890-91), may be expressed at least as briefly in the notations which I have used, and without special devices. But I should not have been satisfied for the purposes of my pamphlet with any notation which should suggest even to the careless reader any connection with the notion of the quaternion. For I confess that one of my objects was to show that a system of vector analysis does not require any support from the notion of the quaternion, or, I may add, of the imaginary in algebra.

I should hardly dare to express myself with so much freedom, if I could not shelter myself behind an authority which will not be questioned.

I do not see that I have done anything very different from what the eminent mathematician upon whom Hamilton's mantle has fallen has been doing, it would seem, unconsciously. Contrast the system of quaternions, which he has described in his sketch of Hamilton's life and work in the *North British Review* for September, 1866, with the system which he urges upon the attention of physicists in the *Philosophical Magazine* in 1890. In 1866 we have a great deal about imaginaries, and nearly as much about the quaternion. In 1890 we have nothing about imaginaries, and little about the quaternion. Prof. Tait has spoken of the calculus of quaternions as throwing off in the course of years its early Cartesian trammels. I wonder that he does not see how well the progress in which he has led may be described as throwing off the yoke of the quaternion. A characteristic example is seen in the use of the symbol ∇. Hamilton applies this to a vector to form a quaternion, Tait to form a linear vector function. But while breathing a new life into the formulæ of quaternions, Prof. Tait stands stoutly by the letter.

Now I appreciate and admire the generous loyalty toward one whom he regards as his master, which has always led Prof. Tait to minimise the originality of his own work in regard to quaternions, and write as if everything was contained in the ideas which flashed into the mind of Hamilton at the classic Brougham Bridge. But not to speak of other claims of historical justice, we owe duties to our scholars as well as to our teachers, and the world is too large, and the current of modern thought is too broad, to be confined by the *ipse dixit* even of a Hamilton.

X.

QUATERNIONS AND VECTOR ANALYSIS.

[*Nature*, vol. XLVIII. pp. 364-367, Aug. 17, 1893.]

In a paper by Prof. C. G. Knott on "Recent Innovations in Vector Theory," of which an abstract has been given in *Nature* (vol. xlvii, pp. 590-593; see also a minor abstract on p. 287), the doctrine that the quaternion affords the only sufficient and proper basis for vector analysis is maintained by arguments based so largely on the faults and deficiencies which the author has found in my pamphlet, *Elements of Vector Analysis*, as to give to such faults an importance which they would not otherwise possess, and to make some reply from me necessary, if I would not discredit the cause of non-quaternionic vector analysis. Especially is this true in view of the warm commendation and endorsement of the paper, by Prof. Tait, which appeared in *Nature* somewhat earlier (p. 225).

The charge which most requires a reply is expressed most distinctly in the minor abstract, viz., "that in the development of his dyadic notation, Prof. Gibbs, being forced to bring the quaternion in, logically condemned his own position." This was incomprehensible to me until I received the original paper, where I found the charge specified as follows: "Although Gibbs gets over a good deal of ground without the explicit recognition of the complete product, which is the difference of his 'skew' and 'direct' products, yet even he recognises in plain language the versorial character of a vector, brings in the quaternion whose vector is the difference of a linear vector function and its conjugate, and does not hesitate to use the accursed thing itself in certain line, surface, and volume integrals" (*Proc. R.S.E.*, Session 1892-3, p. 236). These three specifications I shall consider in their inverse order, premising, however, that the *epitheta ornantia* are entirely my critic's.

The last charge is due entirely to an inadvertence. The integrals referred to are those given at the close of the major abstract in *Nature* (p. 593). My critic, in his original paper, states quite correctly that, according to my definitions and notations, they should represent dyadics. He multiplies them into a vector, introducing the vector under the integral sign, as is perfectly proper, provided, of course, that the vector is constant. But failing to observe this restriction,

evidently through inadvertence, and finding that the resulting equations (thus interpreted) would not be true, he concludes that I must have meant something else by the original equations. Now, these equations will hold if interpreted in the quaternionic sense, as is, indeed, a necessary consequence of their holding in the dyadic sense, although the converse would not be true. My critic was thus led, in consequence of the inadvertence mentioned, to suppose that I had departed from my ordinary usage and my express definitions, and had intended the products in these integrals to be taken in the quaternionic sense. This is the sole ground for the last charge.

The second charge evidently relates to the notations Φ_s and $\Phi_\times$ (see *Nature*, vol. xlvii, p. 592). It is perfectly true that I have used a scalar and a vector connected with the linear vector operator, which, if combined, would form a quaternion. I have not thus combined them. Perhaps Prof. Knott will say that since I use both of them it matters little whether I combine them or not. If so I heartily agree with him.

The first charge is a little vague. I certainly admit that vectors may be used in connection with and to represent rotations I have no objection to calling them in such cases *versorial*. In that sense Lagrange and Poinsot, for example, used versorial vectors. But what has this to do with quaternions? Certainly Lagrange and Poinsot were not quaternionists.

The passage in the major abstract in *Nature* which most distinctly charges me with the use of the quaternion is that in which a certain expression which I use is said to represent the quaternion operator $q(\quad)q^{-1}$ (vol. xlvii, p. 592). It would be more accurate to say that my expression and the quaternionic expression represent the same operator. Does it follow that I have used a quaternion? Not at all. A quaternionic expression may represent a number. Does everyone who uses any expression for that number use quaternions? A quaternionic expression may represent a vector. Does everyone who uses any expression for that vector use quaternions? A quaternionic expression may represent a linear vector operator. If I use an expression for that linear vector operator do I therefore use quaternions? My critic is so anxious to prove that I use quaternions that he uses arguments which would prove that quaternions were in common use before Hamilton was born.

So much for the alleged use of the quaternion in my pamphlet. Let us now consider the faults and deficiencies which have been found therein and attributed to the want of the quaternion. The most serious criticism in this respect relates to certain integrating operators, which Prof. Tait unites with Prof. Knott in ridiculing. As definitions are wearisome, I will illustrate the use of the terms and notations

which I have used by quoting a sentence addressed to the British Association a few years ago. The speaker was Lord Kelvin.

"Helmholtz first solved the problem—Given the spin in any case of liquid motion, to find the motion. His solution consists in finding the potentials of three ideal distributions of gravitational matter having densities respectively equal to $1/\pi$ of the rectangular components of the given spin; and, regarding for a moment these potentials as rectangular components of velocity in a case of liquid motion, taking the spin in this motion as the velocity in the required motion" (*Nature*, vol. xxxviii, p. 569).

In the terms and notations of my pamphlet the problem and solution may be thus expressed:

Given the curl in any case of liquid motion—to find the motion.

The required velocity is $1/4\pi$ of the curl of the potential of the given curl.

Or, more briefly—The required velocity is $\frac{1}{4\pi}$ of the Laplacian of the given curl.

Or in purely analytical form—Required ω in terms of $\nabla\times\omega$, when $\nabla.\omega=0$.

Solution: $\omega=1/4\pi\nabla\times\mathrm{Pot}\,\nabla\times\omega=1/4\pi\,\mathrm{Lap}\,\nabla\times\omega.$

(The Laplacian expresses the result of an operation like that by which magnetic force is calculated from electric currents distributed in space. This corresponds to the second form in which Helmholtz expressed his result.)

To show the incredible rashness of my critics, I will remark that these equations are among those of which it is said in the original paper (*Proc. R.S.E.*, Session 1892-93, p. 225), "Gibbs gives a good many equations—theorems I suppose they ape at being." I may add that others of the equations thus characterized are associated with names not less distinguished than that of Helmholtz. But that to which I wish especially to call attention is that the terms and notations in question express exactly the notions which physicists want to use.

But we are told (*Nature*, vol. xlvii, p. 287) that these integrating operators (Pot, Lap) are best expressed as inverse functions of ∇. To see how utterly inadequate the Nabla would have been to express the idea, we have only to imagine the exclamation points which the members of the British Association would have looked at each other if the distinguished speaker had said:

Helmholtz first solved the problem—Given the Nabla of the velocity in any case of liquid motion, to find the velocity. His solution was that the velocity was the Nabla of the inverse square of Nabla of the Nabla of the velocity. Or, that the velocity was the inverse Nabla of the Nabla of the velocity.

Or, if the problem and solution had been written thus: Required ω in terms of $\nabla\omega$ when $S\nabla\omega = 0$.

Solution: $\omega = \nabla\nabla^{-2}\nabla\omega = \nabla^{-1}\nabla\omega$.

My critic has himself given more than one example of the unfitness of the inverse Nabla for the exact expression of thought. For example, when he says that I have taken "eight distinct steps to prove two equations, which are special cases of

$$\nabla^{-2}\nabla^2 u = u,"$$

I do not quite know what he means. If he means that I have taken eight steps to prove Poisson's Equation (which certainly is not expressed by the equation cited, although it may perhaps be associated with it in some minds), I will only say that my proof is not very long, especially as I have aimed at greater rigor than is usually thought necessary. I cannot, however, compare my demonstration with that of quaternionic writers, as I have not been able (doubtless on account of insufficient search) to find any such.

To show how little foundation there is for the charge that the deficiencies of my system require to be pieced out by these integral operators, I need only say that if I wished to economise operators I might give up New, Lap, and Max, writing for them ∇ Pot, $\nabla\times$Pot, and $\nabla.$Pot, and if I wished further to economise in what costs so little, I could give up the potential also by using the notation $(\nabla.\nabla)^{-1}$ or ∇^{-2}. That is, I could have used this notation without greater sacrifice of precision than quaternionic writers seem to be willing to make. I much prefer, however, to avoid these inverse operators as essentially indefinite.

Nevertheless—although my critic has greatly obscured the subject by ridiculing operators, which I beg leave to maintain are not worthy of ridicule, and by thoughtlessly asserting that it was necessary for me to use them, whereas they are only necessary for me in the sense in which something of the kind is necessary for the quaternionist also, if he would use a notation irreproachable on the score of exactness—I desire to be perfectly candid. I do not wish to deny that the relations connected with these notations appear a little more simple in the quaternionic form. I had, indeed, this subject principally in mind when I said two years ago in *Nature* (vol. xliii, p. 512) [this vol. p. 158]: "There are a few formulæ in which there is a trifling gain in compactness in the use of the quaternion." Let us see exactly how much this advantage amounts to.

There is nothing which the most rigid quaternionist need object to in the notation for the potential, or indeed for the Newtonian. These represent respectively the operations by which the potential or the force of gravitation is calculated from the density of matter. A quaternionist would, however, apply the operator *New* not only to a

scalar, as I have done, but to a vector also. The vector part of New ω (construed in the quaternionic sense) would be exactly what I have represented by Lap ω, and the scalar part, taken negatively, would be exactly what I have represented by Max ω. The quaternionist has here a slight economy in notations, which is of less importance, since all the operators—New, Lap, Max—may be expressed without ambiguity in terms of the potential, which is therefore the only one necessary for the exact expression of thought.

But what are the formulæ which it is necessary for one to remember who uses my notations? Evidently only those which contain the operator *Pot.* For all the others are derived from these by the simple substitutions

$$\begin{aligned} \text{New} &= \nabla \text{ Pot}, \\ \text{Lap} &= \nabla \times \text{Pot}, \\ \text{Max} &= \nabla . \text{Pot}. \end{aligned}$$

Whether one is quaternionist or not, one must remember Poisson's Equation, which I write

$$\nabla . \nabla \text{ Pot } \omega = -4\pi\omega,$$

and in quaternionic might be written

$$\nabla^2 \text{ Pot } \omega = 4\pi\omega.$$

If ω is a vector, in using my equations one has also to remember the general formulæ,

$$\nabla . \nabla\omega = \nabla\nabla . \omega - \nabla \times \nabla \times \omega$$

which as applied to the present case may be united with the preceding in the three-membered equation,

$$\nabla . \nabla \text{ Pot } \omega = \nabla\nabla . \text{ Pot } \omega - \nabla \times \nabla \times \text{Pot } \omega = -4\pi\omega.$$

This single equation is absolutely all that there is to burden the memory of the student, except that the symbols of differentiation (∇, $\nabla\times$, $\nabla.$) may be placed indifferently before or after the symbol for the potential, and that if we choose we may substitute as above *New* for ∇ Pot, etc. Of course this gives a good many equations, which on account of the importance of the subject (as they might almost be said to give the mathematics of the electro-magnetic field) I have written out more in detail than might seem necessary. I have also called the attention of the student to many things, which perhaps he might be left to himself to see. Prof. Knott says that the quaternionist obtains similar equations by the simplest transformations. He has failed to observe that the same is true in my *Vector Analysis*, when once I have proved Poisson's Equation. Perhaps he takes his model of brevity from Prof. Tait, who simplifies the subject, I believe, in his treatise on Quaternions, by taking this theorem for granted.

Nevertheless, since I am forced so often to disagree with Prof. Knott, I am glad to agree with him when I can. He says in his

original paper (p. 226), "No finer argument in favour of the real quaternion vector analysis can be found than in the tangle and the jangle of sections 91 to 104 in the *Elements of Vector Analysis.*" Now I am quite ready to plead guilty to the tangle. The sections mentioned, as is sufficiently evident to the reader, were written at two different times, sections 102-104 being an addition after a couple of years. The matter of these latter sections is not found in its natural place, and the result is well enough characterised as a *tangle.* It certainly does credit to the conscientious study which Prof. Knott has given to my pamphlet, that he has discovered that there is a violent dislocation of ideas just at this point. For such a fault of composition I have no sufficient excuse to offer, but I must protest against its being made the ground of any broad conclusions in regard to the fundamental importance of the quaternion.

Prof. Knott next proceeds to criticise—or at least to ridicule—my treatment of the linear vector function, with respect to which we read in the abstract:—"As developed in the pamphlet, the theory of the dyadic goes over much the same ground as is traversed in the last chapter of Kelland and Tait's *Introduction to Quaternions.* With the exception of a few of those lexicon products, for which Prof. Gibbs has such an affection, there is nothing of real value added to our knowledge of the linear vector function." It would not, I think, be difficult to show some inaccuracy in my critic's characterisation of the real content of this part of my pamphlet. But as algebra is a formal science, and as the whole discussion is concerning the best form of representing certain kinds of relations, the important question would seem to be whether there is anything of *formal* value in my treatment of the linear vector function.

Now Prof. Knott distinctly characterises in half a dozen words the difference in the spirit and method of my treatment of this subject from that which is traditional among quaternionists, when he says of what I have called dyadics—"these are not quantities, but operators" (*Nature*, vol. xlvii, p. 592). I do not think that I applied the word quantity to the dyadics, but Prof. Knott recognised that I treated them as quantities—not, of course, as the quantities of arithmetic, or of ordinary algebra, but as quantities in the broader sense, in which, for example, quaternions are called quantities. The fact that they may be operators does not prevent this. Just as in grammar verbs may be taken as substantives, viz., in the infinitive mood, so in algebra operators—especially such as are capable of quantitative variation—may be regarded as quantities when they are made the subject of algebraic comparison or operation. Now I would not say that it is necessary to treat every kind of operator as quantity, but I certainly think that one so important as the linear vector operator, and one

which lends itself so well to such broader treatment, is worthy of it. Of course, when vectors are treated by the methods of ordinary algebra, linear vector operators will naturally be treated by the same methods, but in an algebra formed for the sake of expressing the relations between vectors, and in which vectors are treated as multiple quantities, it would seem an incongruity not to apply the methods of multiple algebra also to the linear vector operator.

The dyadic is practically the linear vector operator regarded as quantity. More exactly it is the multiple quantity of the ninth order which affords various operators according to the way in which it is applied. I will not venture to say what ought to be included in a treatise on quaternions, in which, of course, a good many subjects would have claims prior to the linear vector operator; but for the purposes of my pamphlet, in which the linear vector operator is one of the most important topics, I cannot but regard a treatment like that in Hamilton's *Lectures*, or *Elements*, as wholly inadequate *on the formal side.* To show what I mean, I have only to compare Hamilton's treatment of the quaternion and of the linear vector operator with respect to notations. Since quaternions have been identified with matrices, while the linear vector operator evidently belongs to that class of multiple quantities, it seems unreasonable to refuse to the one those notations which we grant to the other. Thus, if the quaternionist has e^q, $\log q$, $\sin q$, $\cos q$, why should not the vector analyst have e^Φ, $\log \Phi$, $\sin \Phi$, $\cos \Phi$, where Φ represents a linear vector operator? I suppose the latter are at least as useful to the physicist. I mention these notations first, because here the analogy is most evident. But there are other cases far more important, because more elementary, in which the analogy is not so near the surface, and therefore the difference in Hamilton's treatment of the two kinds of multiple quantity not so evident. We have, for example, the tensor of the quaternion, which has the important property represented by the equation: $\mathrm{T}(qr) = \mathrm{T}q\mathrm{T}r$.

There is a scalar quantity related to the linear vector operator, which I have represented by the notation $|\Phi|$ and called the *determinant* of Φ. It is in fact the determinant of the matrix by which Φ may be represented, just as the square of the tensor of q (sometimes called the *norm* of q) is the determinant of the matrix by which q may be represented. It may also be defined as the product of the latent roots of Φ, just as the square of the tensor of q might be defined as the product of the latent roots of q. Again, it has the property represented by the equation

$$|\Phi.\Psi| = |\Phi||\Psi|$$

which corresponds exactly with the preceding equation with both sides squared.

There is another scalar quantity connected with the quaternion and represented by the notation Sq. It has the important property expressed by the equation,

$$S(qrs)=S(rsq)=S(sqr),$$

and so for products of any number of quaternions, in which the cyclic order remains unchanged. In the theory of the linear vector operator there is an important quantity which I have represented by the notation Φ_S, and which has the property represented by the equation

$$(\Phi.\Psi.\Omega)_S=(\Psi.\Omega.\Phi)_S=(\Omega.\Phi.\Psi)_S,$$

where the number of the factors is as before immaterial. Φ_S may be defined as the sum of the latent roots of Φ, just as $2Sq$ may be defined as the sum of the latent roots of q.

The analogy of these notations may be further illustrated by comparing the equations

$$T(e^q)=e^{Sq}$$

and

$$|e^{\Phi}|=e^{\Phi_S}.$$

I do not see why it is not as reasonable for the vector analyst to have notations like $|\Phi|$ and Φ_S, as for the quaternionist to have the notations Tq and Sq.

This is of course an *argumentum ad quaternionisten*. I do not pretend that it gives the reason why I used these notations, for the identification of the quaternion with a matrix was, I think, unknown to me when I wrote my pamphlet. The real justification of the notations $|\Phi|$ and Φ_S is that they express functions of the linear vector operator *quâ* quantity, which physicists and others have continually occasion to use. And this justification applies to other notations which may not have their analogues in quaternions. Thus I have used $\Phi_\times$ to express a vector so important in the theory of the linear vector operator, that it can hardly be neglected in any treatment of the subject. It is described, for example, in treatises as different as Thomson and Tait's *Natural Philosophy* and Kelland and Tait's *Quaternions*. In the former treatise the components of the vector are, of course, given in terms of the elements of the linear vector operator, which is in accordance with the method of the treatise. In the latter treatise the vector is expressed by

$$V\alpha\alpha'+V\beta\beta'+V\gamma\gamma'.$$

As this supposes the linear vector operator to be given not by a single letter, but by several vectors, it must be regarded as entirely inadequate by any one who wishes to treat the subject in the spirit of multiple algebra, *i.e.* to use a single letter to represent the linear vector operator.

But my critic does not like the notations $|\Phi|$, Φ_S, $\Phi_\times$. His ridicule,

indeed réaches high-water mark in the paragraphs in which he mentions them. Concerning another notation, $\Phi^{\times}_{\times}\Phi$ (defined in *Nature*, vol. xliii, p. 513) [this vol., p. 160], he exclaims, "Thus burden after burden, in the form of new notation, is added apparently for the sole purpose of exercising the faculty of memory." He would vastly prefer, it would appear, to write with Hamilton $m\phi'^{-1}$, "where m represents what the unit volume becomes under the influence of the linear operator." But this notation is only apparently compact, since the m requires explanation. Moreover, if a strain were given in what Hamilton calls the standard trinomial form, to write out the formula for the operator on surfaces in that standard form by the use of the expression $m\phi'^{-1}$ would require, it seems to me, ten (if not fifty) times the effort of memory and of ingenuity, which would be required for the same purpose with the use of $\frac{1}{2}\Phi^{\times}_{\times}\Phi$.

I may here remark that Prof. Tait's letter of endorsement of Prof. Knott's paper affords a striking illustration of the convenience and flexibility of a notation entirely analogous to $\Phi^{\times}_{\times}\Phi$, viz., $\Phi{:}\Phi$. He gives the form $S\nabla\nabla_1\, S\sigma\sigma_1$ to illustrate the advantage of quaternionic notations in point of brevity. If I understand his notation, this is what I should write $\nabla\sigma{:}\nabla\sigma$. (I take for granted that the suffixes indicate that ∇ applies as differential operator to σ, and ∇_1 to σ_1, σ and σ_1 being really identical in meaning, as also ∇ and ∇_1.) It will be observed that in my notation one dot unites in multiplication the two ∇'s, and the other the two σ's, and that I am able to leave each ∇ where it naturally belongs as differential operator. The quaternionist cannot do this, because the ∇ and σ cannot be left together without uniting to form a quaternion, which is not at all wanted. Moreover, I can write Φ for $\nabla\sigma$, and $\Phi{:}\Phi$ for $\nabla\sigma{:}\nabla\sigma$. The quaternionist also uses a ϕ, which is practically identical with my Φ (viz., the operator which expresses the relation between $d\sigma$ and $d\rho$), but I do not see how Prof. Knott, who I suppose dislikes $\Phi{:}\Phi$ as much as $\Phi^{\times}_{\times}\Phi$, would express $S\nabla\nabla_1\, S\sigma\sigma_1$ in terms of this ϕ.

It is characteristic of Prof. Knott's view of the subject, that in translating into quaternionic from a dyadic, or operator, as he calls it, he adds in each case an operand. In many cases it would be difficult to make the translation without this. But it is often a distinct advantage to be able to give the operator without the operand. For example, in translating into quaternionic my dyadic or operator $\Phi\times\rho$, he adds an operand, and exclaims, "The old thing!" Certainly, when this expression is applied to an operand, there is no advantage (and no disadvantage) in my notation as compared with the quaternionic. But if the quaternionist wished to express what I would write in the form $(\Phi\times\rho)^{-1}$, or $|\Phi\times\rho|$, or $(\Phi\times\rho)_S$, or $(\Phi\times\rho)_\times$, he would, I think find the operand very much in the way.

XI.

ON DOUBLE REFRACTION AND THE DISPERSION OF COLORS IN PERFECTLY TRANSPARENT MEDIA.

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1. In calculating the velocity of a system of plane waves of homogeneous light, regarded as oscillating electrical fluxes, in transparent and sensibly homogeneous bodies, whether singly or doubly refracting, we may assume that such a body is a very fine-grained structure, so that it can be divided into parts having their dimensions very small in comparison with the wave-length, each of which may be regarded as entirely similar to every other, while in the interior of each there are wide differences in electrical as in other physical properties. Hence, the average electrical displacement in such parts of the body may be expressed as a function of the time and the coordinates of position by the ordinary equations of wave-motion, while the real displacement at any point will in general differ greatly from that represented by such equations.

It is the object of this paper to investigate the velocity of light in perfectly transparent media which have not the property of circular polarization in a manner which shall take account of this difference between the real displacements and those represented by the ordinary equations of wave-motion. We shall find that this difference will account for the dispersion of colors, without affecting the validity of the laws of Huyghens and Fresnel for double refraction with respect to light of any one color.

In this investigation, it is assumed that the electrical displacements are *solenoidal*, or, in other words, that they are such as not to produce any change in electrical density. The disturbance in the medium is treated as consisting entirely of such electrical displacements and fluxes, and not complicated by any distinctively magnetic phenomena. It might therefore be more accurate to call the theory (as here developed) *electrical* rather than *electromagnetic*. The latter term is nevertheless retained in accordance with general usage, and with that of the author of the theory.

Since the velocity which we are seeking is equal to the wave-length divided by the period of oscillation, the problem reduces to finding the ratio of these quantities, and may be simplified in some respects

by supposing that we have to do with a system of stationary waves. That the relation of the wave-length and the period is the same for stationary as for progressive waves is evident from the consideration that a system of stationary waves may be formed by two systems of progressive waves having opposite directions.

2. Let x, y, z be the rectangular coordinates of any point in the medium, which with the system of waves we may regard as indefinitely extended, and let $\xi+\xi'$, $\eta+\eta'$, $\zeta+\zeta'$ be the components of electrical displacement at that point at the time t; ξ, η, ζ being the average values of the components of electrical displacement at that time in a wave-plane passing through the point. Then $\xi, \eta, \zeta, \xi', \eta', \zeta'$ are perfectly defined quantities, of which ξ, η, ζ are connected with x, y, z, and t by the ordinary equations of wave-motion, while each of the quantities ξ', η', ζ' has always zero for its average value in any wave-plane. We may call ξ, η, ζ the components of the *regular* part of the displacement, and ξ', η', ζ' the components of the *irregular* part of the displacement. In like manner, the differential coefficients of these quantities with respect to the time, $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, $\dot{\xi}'$, $\dot{\eta}'$, $\dot{\zeta}'$, may be called respectively the components of the regular part of the flux, and the components of the irregular part of the flux.

Let the whole space be divided into elements of volume Dv, very small in all dimensions in comparison with a wave-length, but enclosing portions of the medium which may be treated as entirely similar to one another, and therefore not infinitely small. Thus a crystal may be divided into elementary parallelopipeds, all the vertices of which are similarly situated with respect to the internal structure of the crystal. Amorphous solids and liquids may not be capable of division into equally small portions of which physical similarity can be predicated with the same rigor. Yet we may suppose them capable of a division substantially satisfying the requirements.

From these definitions it follows that at any given instant the average value of each of the quantities ξ', η', ζ' in an element Dv is zero. For the average value in one such element must be sensibly the same as in any other situated on the same wave-plane. If this average were not zero, the average for the wave-plane would not be zero. Moreover, at any given instant, the values of ξ, η, ζ may be regarded as constant throughout any element Dv, and as representing the average values of the components of displacement in that element. The same will be true of the quantities $\dot{\xi}'$, $\dot{\eta}'$, $\dot{\zeta}'$ and $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$.

3. Since we have excluded the case of media which have the property of circular polarization, we shall not impair the generality of our results if we suppose that we have to do with linearly polarized light, i.e., that the regular part of the displacement is everywhere parallel to the same fixed line, all cases not already excluded being

reducible to this. Then, with the origin of coordinates and the zero of time suitably chosen, the regular part of the displacement may be represented by the equations

$$\left.\begin{aligned} \xi &= \alpha \cos 2\pi \frac{u}{l} \cos 2\pi \frac{t}{p}, \\ \eta &= \beta \cos 2\pi \frac{u}{l} \cos 2\pi \frac{t}{p}, \\ \zeta &= \gamma \cos 2\pi \frac{u}{l} \cos 2\pi \frac{t}{p}, \end{aligned}\right\} \qquad (1)$$

where l denotes the wave-length, p the period of vibration, α, β, γ the maximum amplitudes of the displacements ξ, η, ζ, and u the distance of the point considered from the wave-plane which passes through the origin. Since u is a linear function of x, y, and z, we may regard these equations as giving the values of ξ, η, ζ, for a given system of waves, in terms of x, y, z, and t.

4. The components of the irregular displacement, ξ', η', ζ', at any given point, will evidently be simple harmonic functions of the time, having the same period as the regular part of the displacement. That they will also have the same phase is not quite so evident, and would not be the case in a medium in which there were any absorption or dispersion of light. It will however appear from the following considerations that in perfectly transparent media the irregular oscillations are synchronous with the regular. For if they are not synchronous, we may resolve the irregular oscillations into two parts, of which one shall be synchronous with the regular oscillations, and the other shall have a difference of phase of one-fourth of a complete oscillation. Now if the medium is one in which there is no absorption or dispersion of light, we may assume that the same electrical configurations may also be passed through in the inverse order, which would be represented analytically by writing $-t$ for t in the equations which give ξ, η, ζ, ξ', η', ζ', as functions of x, y, z, and t. But this change would not affect the regular oscillations, nor the synchronous part of the irregular oscillations, which depends on the cosine of the time, while the non-synchronous part of the irregular oscillations, which depends on the sine of the time, would simply have its direction reversed. Hence, by taking first one-half the sum, and secondly one-half the difference, of the original motion and that obtained by substitution of $-t$ for t, we may separate the non-synchronous part of the irregular oscillations from the rest of the motion. Therefore, the supposed non-synchronous part of the irregular displacement, if capable of existence, is at least wholly independent of the wave-motion and need not be considered by us.

We may go farther in the determination of the quantities ξ', η', ζ'. For in view of the very fine-grained structure of the medium, it will

easily appear that the manner in which the general or average flux in any element Dv (represented by ξ, η, ζ) distributes itself among the molecules and intermolecular spaces must be entirely determined by the amount and direction of that flux and its period of oscillation. Hence, and on account of the superposable character of the motions which we are considering, we may conclude that the values of ξ', η', ζ' at any given point in the medium are capable of expression as linear functions of ξ, η, ζ in a manner which shall be independent of the time and of the orientation of the wave-planes and the distance of a nodal plane from the point considered, so long as the period of oscillation remains the same. But a change in the period may presumably affect the relation between ξ', η', ζ' and ξ, η, ζ to a certain extent. And the relation between ξ', η', ζ' and ξ, η, ζ will vary rapidly as we pass from one point to another within the element Dv.

5. In the motion which we are considering there occur alternately instants of no velocity and instants of no displacement. The statical energy of the medium at an instant of no velocity must be equal to its kinetic energy at an instant of no displacement. Let us examine each of these quantities, and consider the equation which expresses their equality.

6. Since in every part of an element Dv the irregular as well as the regular part of the displacement is entirely determined (for light of a given period) by the values of ξ, η, ζ, the statical energy of the element must be a quadratic function of ξ, η, ζ, say

$$(A\xi^2+B\eta^2+C\zeta^2+E\eta\zeta+F\zeta\xi+G\xi\eta)Dv,$$

where A, B, etc. depend only on the nature of the medium and the period of oscillation. At an instant of no velocity, when

$$\sin 2\pi\frac{t}{p}=0, \quad \text{and} \quad \cos^2 2\pi\frac{t}{p}=1,$$

the above expression will reduce by equations (1) to

$$(A\alpha^2+B\beta^2+C\gamma^2+E\beta\gamma+F\gamma\alpha+G\alpha\beta)\cos^2 2\pi\frac{u}{l}Dv.$$

Since the average value of $\cos^2 2\pi\frac{u}{l}$ in an indefinitely extended space is $\frac{1}{2}$, we have for the statical energy in a unit of volume

$$S=\tfrac{1}{2}(A\alpha^2+B\beta^2+C\gamma^2+E\beta\gamma+F\gamma\alpha+G\alpha\beta). \qquad (2)$$

7. The kinetic energy of the whole medium is represented by the double volume-integral*

$$\tfrac{1}{2}\Sigma\iint\frac{(\dot{\xi}+\dot{\xi}')_1(\dot{\xi}+\dot{\xi}')_2}{r}dv_1\,dv_2,$$

* The fluxes are supposed to be measured by the electromagnetic system of units. It is to be observed that the difference of opinion which has prevailed with respect to the estimation of the energy of electrical currents does not extend to such as are *solenoidal*, which may be regarded as composed of closed circuits.

where dv_1, dv_2 are two infinitesimal elements of volume, $(\dot{\xi}+\dot{\xi}')_1$, $(\dot{\xi}+\dot{\xi}')_2$ the corresponding components of flux, r the distance between the elements, and Σ denotes a summation with respect to the coordinate axes. Separating the integrations, we may write for the same quantity

$$\tfrac{1}{2}\Sigma\int(\dot{\xi}+\dot{\xi}')_1\Big[\int\frac{(\dot{\xi}+\dot{\xi}')_2}{r}dv_2\Big]dv_1.$$

It is evident that the integral within the brackets is derived from $\dot{\xi}+\dot{\xi}'$ by the same process by which the potential of any mass is derived from its density. If we use the symbol *Pot* to express this relation, we may write for the kinetic energy

$$\tfrac{1}{2}\Sigma\int(\dot{\xi}+\dot{\xi}')\,\mathrm{Pot}\,(\dot{\xi}+\dot{\xi}')dv.$$

The operation denoted by this symbol is evidently distributive, so that $\mathrm{Pot}\,(\dot{\xi}+\dot{\xi}')=\mathrm{Pot}\,\dot{\xi}+\mathrm{Pot}\,\dot{\xi}'$. The expression for the kinetic energy may therefore be expanded into

$$\tfrac{1}{2}\Sigma\int\dot{\xi}\,\mathrm{Pot}\,\dot{\xi}\,dv+\tfrac{1}{2}\Sigma\int\dot{\xi}\,\mathrm{Pot}\,\dot{\xi}'\,dv+\tfrac{1}{2}\Sigma\int\dot{\xi}'\,\mathrm{Pot}\,\dot{\xi}\,dv+\tfrac{1}{2}\Sigma\int\dot{\xi}'\,\mathrm{Pot}\,\dot{\xi}'\,dv.$$

But $\dot{\xi}'$, and therefore $\mathrm{Pot}\,\dot{\xi}'$, has in every wave-plane the average value zero. Also $\dot{\xi}$, and therefore $\mathrm{Pot}\,\dot{\xi}$, has in every wave-plane a constant value. Therefore the second and third integrals in the above expression will vanish, leaving for the kinetic energy

$$\tfrac{1}{2}\Sigma\int\dot{\xi}\,\mathrm{Pot}\,\dot{\xi}\,dv+\tfrac{1}{2}\Sigma\int\dot{\xi}'\,\mathrm{Pot}\,\dot{\xi}'\,dv, \tag{3}$$

which is to be calculated for a time of no displacement, when

$$\dot{\xi}=\pm\frac{2\pi a}{p}\cos 2\pi\frac{u}{l},\quad \dot{\eta}=\pm\frac{2\pi\beta}{p}\cos 2\pi\frac{u}{l},\quad \dot{\zeta}=\pm\frac{2\pi\gamma}{p}\cos 2\pi\frac{u}{l}. \tag{4}$$

The form of the expression (3) indicates that the kinetic energy consists of two parts, one of which is determined by the regular part of the flux, and the other by the irregular part of the flux.

8. The value of $\mathrm{Pot}\,\dot{\xi}$ may be easily found by integration, but perhaps more readily by Poisson's well-known theorem, that if q is any function of position in space (as the density of a certain mass),

$$\frac{d^2\,\mathrm{Pot}\,q}{dx^2}+\frac{d^2\,\mathrm{Pot}\,q}{dy^2}+\frac{d^2\,\mathrm{Pot}\,q}{dz^2}=-4\pi q, \tag{5}$$

where the direction of the coordinate axes is immaterial, provided that they are rectangular. In applying this to $\mathrm{Pot}\,\dot{\xi}$, we may place two of the axes in a wave-plane. This will give

$$\frac{d^2\,\mathrm{Pot}\,\dot{\xi}}{du^2}=-4\pi\dot{\xi}. \tag{6}$$

In a nodal plane, $\mathrm{Pot}\,\dot{\xi}=0$, since $\dot{\xi}$ has equal positive and negative values in elements of volume symmetrically distributed with respect

to any point in such a plane. In a wave-crest (or plane in which $\dot{\xi}$ has a maximum value), Pot $\dot{\xi}$ will also have a maximum value, which we may call K. For intermediate points we may determine its value from the consideration that the total disturbance may be resolved into two systems of waves, one having a wave-crest, and the other a nodal plane passing through the point for which the potential is sought. The maximum amplitudes of these component systems will be to the maximum amplitude of the original system as $\cos 2\pi\frac{u}{l}$ and $\sin 2\pi\frac{u}{l}$ to unity. But the second of the component systems will contribute nothing to the value of the potential. We thus obtain

$$\text{Pot } \dot{\xi} = \text{K} \cos 2\pi\frac{w}{l},$$

$$\frac{d^2 \text{ Pot } \dot{\xi}}{du^2} = -\frac{4\pi^2}{l^2}\text{K} \cos 2\pi\frac{u}{l} = -\frac{4\pi^2}{l^2}\text{ Pot } \dot{\xi}.$$

Comparing this with equation (6), we have

$$-\frac{4\pi^2}{l^2}\text{ Pot } \dot{\xi} = -4\pi\dot{\xi},$$

$$\text{Pot } \dot{\xi} = \frac{l^2}{\pi}\dot{\xi}. \tag{7}$$

Hence, and by equations (4),

$$\tfrac{1}{2}\Sigma\int\dot{\xi}\text{ Pot } \dot{\xi}\,dv = \frac{l^2}{2\pi}\Sigma\int\dot{\xi}^2\,dv = \frac{2\pi l^2}{p^2}(\alpha^2+\beta^2+\gamma^2)\int\cos^2 2\pi\frac{u}{l}\,dv.$$

The kinetic energy of the regular part of the flux is therefore, for each unit of volume,

$$\text{T} = \frac{\pi l^2}{p^2}(\alpha^2+\beta^2+\gamma^2). \tag{8}$$

9. With respect to the kinetic energy of the irregular part of the flux, it is to be observed that, since $\dot{\xi}'$, $\dot{\eta}'$, $\dot{\zeta}'$ have their average values zero in spaces which are very small in comparison with a wave-length, the integrations implied in the notations Pot $\dot{\xi}'$, Pot $\dot{\eta}'$, Pot $\dot{\zeta}'$ may be confined to a sphere of a radius which is small in comparison with a wave-length. Since within such a sphere $\dot{\xi}'$, $\dot{\eta}'$, $\dot{\zeta}'$ are sensibly determined by the values of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ at the center of the sphere, which is the point for which the value of the potentials are sought, Pot $\dot{\xi}'$, Pot $\dot{\eta}'$, Pot $\dot{\zeta}'$ must be functions—evidently linear functions—of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$; and $\dot{\xi}'$ Pot $\dot{\xi}'$, $\dot{\eta}'$ Pot $\dot{\eta}'$, $\dot{\zeta}'$ Pot $\dot{\zeta}'$ must be quadratic functions of the same quantities. But these functions will vary with the position of the point considered with reference to the adjacent molecules.

Now the expression for the kinetic energy of the irregular part of the flux,

$$\tfrac{1}{2}\Sigma\int\dot{\xi}'\text{ Pot } \dot{\xi}'\,dv,$$

indicates that we may regard the infinitesimal element dv as having the energy (due to this part of the flux)

$$\tfrac{1}{2}\Sigma\,\dot{\xi}'\,\mathrm{Pot}\,\dot{\xi}'\,dv.$$

Let us consider the energy due to the irregular flux which will belong to the above defined element Dv, which is not infinitely small, but which has the advantage of being one of physically similar elements which make up the whole medium. The energy of this element is found by adding the energies of all the infinitesimal elements of which it is composed. Since these are quadratic functions of the quantities $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, which are sensibly constant throughout the element Dv, the sum will be a quadratic function of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, say

$$(A'\dot{\xi}^2+B'\dot{\eta}^2+C'\dot{\zeta}^2+E'\dot{\eta}\dot{\zeta}+F'\dot{\zeta}\dot{\xi}+G'\dot{\xi}\dot{\eta})Dv,$$

which will therefore represent the energy of the element Dv due to the irregular flux. The coefficients A′, B′, etc., are determined by the nature of the medium and the period of oscillation. They will be constant throughout the medium, since one element Dv does not differ from another.

This expression reduces by equations (4) to

$$\frac{4\pi^2}{p^2}(A'\alpha^2+B'\beta^2+C'\gamma^2+E'\beta\gamma+F'\gamma\alpha+G'\alpha\beta)\cos^2 2\pi\frac{u}{l}\,Dv.$$

The kinetic energy of the irregular flux in a unit of volume is therefore

$$T'=\frac{2\pi^2}{p^2}(A'\alpha^2+B'\beta^2+C'\gamma^2+E'\beta\gamma+F'\gamma\alpha+G'\alpha\beta). \tag{9}$$

10. Equating the statical and kinetic energies, we have

$$\tfrac{1}{2}(A\alpha^2+B\beta^2+C\gamma^2+E\beta\gamma+F\gamma\alpha+G\alpha\beta)$$
$$=\frac{\pi l^2}{p^2}(\alpha^2+\beta^2+\gamma^2)+\frac{2\pi^2}{p^2}(A'\alpha^2+B'\beta^2+C'\gamma^2+E'\beta\gamma+F'\gamma\alpha+G'\alpha\beta). \tag{10}$$

The velocity (V) of the corresponding system of progressive waves is given by the equation

$$V^2=\frac{l^2}{p^2}=\frac{1}{2\pi}\,\frac{A\alpha^2+B\beta^2+C\gamma^2+E\beta\gamma+F\gamma\alpha+G\alpha\beta}{\alpha^2+\beta^2+\gamma^2}$$
$$-\frac{2\pi}{p^2}\,\frac{A'\alpha^2+B'\beta^2+C'\gamma^2+E'\beta\gamma+F'\gamma\alpha+G'\alpha\beta}{\alpha^2+\beta^2+\gamma^2}. \tag{11}$$

If we set

$$a=\frac{1}{2\pi}A-\frac{2\pi}{p^2}A', \quad b=\frac{1}{2\pi}B-\frac{2\pi}{p^2}B', \quad \text{etc.,} \tag{12}$$

and

$$\rho^2=\alpha^2+\beta^2+\gamma^2,$$

the equation reduces to

$$V^2=\frac{a\alpha^2+b\beta^2+c\gamma^2+e\beta\gamma+f\gamma\alpha+g\alpha\beta}{\rho^2}. \tag{13}$$

For a given medium and light of a given period, the coefficients a, b, etc., are constant.

This relation between the velocity of the waves and the direction of oscillation is capable of a very simple geometrical expression. Let r be the radius vector of the ellipsoid

$$ax^2+by^2+cz^2+eyz+fzx+gxy=1. \tag{14}$$

Then

$$\frac{1}{r^2}=\frac{ax^2+by^2+cz^2+eyz+fzx+gxy}{r^2}.$$

If this radius is drawn parallel to the electrical oscillations, we shall have

$$\frac{x}{r}=\frac{\alpha}{\rho}, \quad \frac{y}{r}=\frac{\beta}{\rho}, \quad \frac{z}{r}=\frac{\gamma}{\rho},$$

and

$$V^2=\frac{1}{r^2}. \tag{15}$$

That is, the wave-velocity for any particular direction of oscillation is represented in the ellipsoid by the reciprocal of the radius vector which is parallel to that direction.

11. This relation between the wave-length, the period, and the direction of vibration, must hold true not only of such vibrations as actually occur, but also of such as we may imagine to occur under the influence of constraints determining the direction of vibration in the wave-plane. The directions of the natural or unconstrained vibrations in any wave-plane may be determined by the general mechanical principle that if the type of a natural vibration is infinitesimally altered by the application of a constraint, the value of the period will be stationary.* Hence, in a system of stationary waves such as we have been considering, if the direction of an unconstrained vibration is infinitesimally varied in its wave-plane by a constraint while the wave-length remains constant, the period will be stationary. Therefore, if the direction of the unconstrained vibration is infinitesimally varied by constraint, and the period remains rigorously constant, the wave-length will be stationary. Hence, if we make a central section of the above described ellipsoid parallel to any wave-plane, the directions of natural vibration for that wave-plane will be parallel to the radii vectores of stationary value in that section, viz., to the axes of the ellipse, when the section is elliptical, or to all radii, when the section is circular.

12. For light of a single period, our hypothesis has led to a perfectly definite result, our equations expressing the fundamental laws of double refraction as enunciated by Fresnel. But if we ask how the velocity of light varies with the period, that is, if we seek

* See Rayleigh's *Theory of Sound*, vol. i, p. 84.

to derive from the same equations the laws of the dispersion of colors, we shall not be able to obtain an equally definite result, since the quantities A, B, etc., and A′, B′, etc., are unknown functions of the period. If, however, we make the assumption, which is hardly likely to be strictly accurate, but which may quite conceivably be not far removed from the truth, that the manner in which the general or average flux in any small part of the medium distributes itself among the molecules and intermolecular spaces is independent of the period, the quantities A, B, etc., and A′, B′, etc., will be constant, and we obtain a very simple relation between V and p, which appears to agree tolerably well with the results of experiment.

If we set $$H=\frac{A\alpha^2+B\beta^2+C\gamma^2+E\beta\gamma+F\gamma\alpha+G\alpha\beta}{\rho^2}, \tag{16}$$

and $$H'=\frac{A'\alpha^2+B'\beta^2+C'\gamma^2+E'\beta\gamma+F'\gamma\alpha+G'\alpha\beta}{\rho^2}, \tag{17}$$

our general equation (11) becomes

$$V^2=\frac{H}{2\pi}-\frac{2\pi H'}{p^2}, \tag{18}$$

where H and H′ will be constant for any given direction of oscillation, when A, B, etc., and A′, B′, etc., are constant. If we wish to introduce into the equation the absolute index of refraction (n) and the wave-length in vacuo (λ) in place of V and p, we may divide both sides of the equation by the square of the constant (k) representing the velocity of light in vacuo. Then, since

$$\frac{V}{k}=\frac{1}{n}, \text{ and } kp=\lambda,$$

our equation reduces to

$$\frac{1}{n^2}=\frac{H}{2\pi k^2}-\frac{2\pi H'}{\lambda^2}. \tag{19}$$

It is well known that the relation between n and λ may be tolerably well but by no means perfectly represented by an equation of this form.

13. If we now give up the presumably inaccurate supposition that A, B, etc., and A′, B′, etc., are constant, equation (19) will still subsist, but H and H′ will not be constant for a given direction of oscillation, but will be functions of p, or, what amounts to the same, of λ. Although we cannot therefore use the equation to derive *a priori* the relation between n and λ, we may use it to derive the values of H and H′ from the empirically determined relation between n and λ. To do this, we must make use again of the general principle that an

infinitesimal variation in the type of a vibration, due to a constraint, will not affect the period. If we first consider a certain system of stationary waves, then a system in which the wave-length is greater by an infinitesimal dl (the direction of oscillation remaining the same), the period will be increased by an infinitesimal dp, and the manner in which the flux distributes itself among the molecules and intermolecular spaces will presumably be infinitesimally changed. But if we suppose that in the second system of waves there is applied a constraint compelling the flux to distribute itself in the same way among the molecules and intermolecular spaces as in the first system (so that ξ', η', ζ' shall be the same functions as before of ξ, η, ζ,—a supposition perfectly compatible with the fact that the values of ξ, η, ζ are changed), this constraint, according to the principle cited, will not affect the period of oscillation. Our equations will apply to such a constrained type of oscillation, and A, B, etc., and A′, B′, etc., and therefore H and H′, will have the same values in the last described system of waves as in the first system, although the wave-length and the period have been varied. Therefore, in differentiating equation (18), which is essentially an equation between l and p, or its equivalent (19), we may treat H and H′ as constant. This gives

$$-\frac{2}{n^3}\frac{dn}{d\lambda}=\frac{4\pi H'}{\lambda^3}.$$

We thus obtain the values of H′ and H

$$H'=-\frac{\lambda^3}{2\pi n^3}\frac{dn}{d\lambda}, \qquad H=\frac{2\pi k^2}{n^2}-\frac{2\pi k^3\lambda}{n^3}\frac{dn}{d\lambda}. \tag{20}$$

By determining the values of H and H′ for different directions of oscillation, we may determine the values of A, B, etc., and A′, B′, etc.

By means of these equations, the ratios of the statical energy (S), the kinetic energy due to the regular part of the flux (T), and the kinetic energy due to the irregular part of the flux (T′), are easily obtained in a form which admits of experimental determination. Equations (8) and (9) give

$$T=\frac{\pi\rho^2 l^2}{p^2}, \qquad T'=\frac{2\pi^2 H'\rho^2}{p^2}.$$

Therefore, by (20),

$$\frac{T'}{T}=\frac{2\pi H'}{l^2}=\frac{2\pi H' n^2}{\lambda^2}=-\frac{\lambda}{n}\frac{dn}{d\lambda}=-\frac{d\log n}{d\log\lambda}. \tag{21}$$

$$\frac{S}{T}=\frac{T+T'}{T}=1+\frac{T'}{T}=\frac{d\log\lambda-d\log n}{d\log\lambda}=\frac{d\log l}{d\log\lambda}. \tag{22}$$

$$\frac{T'}{S}=-\frac{d\log n}{d\log l}. \tag{23}$$

Since S, T, and T′ are essentially positive quantities, their ratios must be positive. Equation (21) therefore requires that the index of refraction shall increase as the period or wave-length in vacuo diminishes. Experiment has shown no exceptions to this rule, except such as are manifestly attributable to the absorption of light.

14. It remains to consider the relations between the optical properties of a medium and the planes or axes of symmetry which it may possess. If we consider the statical energy per unit of volume (S) and the period as constant, we may regard equation (2) as the equation of an ellipsoid, the radii vectores of which represent in direction and magnitude the amplitudes of systems of waves having the same statical energy. In like manner, if we consider the kinetic energy of the irregular part of the flux per unit of volume (T′) and the period as constant, we may regard equation (9) as the equation of an ellipsoid, the radii vectores of which represent in direction and magnitude the amplitudes of systems of waves having the same kinetic energy due to the irregular part of the flux. These ellipsoids, which we may distinguish as the ellipsoids (A, B, etc.) and (A′, B′, etc.), as well as the ellipsoid before described, which we may call the ellipsoid (a, b, etc.), must be independent in their form and their orientation of the directions of the axes of coordinates, being determined entirely by the nature of the medium and the period of oscillation. They must therefore possess the same kind of symmetry as the internal structure of the medium.

If the medium is symmetrical about a certain axis, each ellipsoid must have an axis parallel to that. If the medium is symmetrical with respect to a certain plane, each ellipsoid must have an axis at right angles to that plane. If the medium after a revolution of less than 180° about a certain axis is then equivalent to the medium in its first position, or symmetrical with it with respect to a plane at right angles to that axis, each ellipsoid must have an axis of revolution parallel to that axis. These relations must be the same for light of all colors, and also for all temperatures of the medium.

15. From these principles we may infer the optical characteristics of the different crystallographic systems.

In crystals of the isometric system, as in amorphous bodies, the three ellipsoids reduce to spheres. Such media are optically isotropic at least so far as any properties are concerned which come within the scope of this paper.

In crystals of the tetragonal or hexagonal systems, the three ellipsoids will have axes of rotation parallel to the principal crystallographic axis. Since the ellipsoid (a, b, etc.) has but one circular section, there will be but one optic axis, which will have a fixed direction.

In crystals of the orthorhombic system, the three ellipsoids will have their axes parallel to the rectangular crystallographic axes. If we take these directions for the axes of coordinates, E, F, G, E′, F′, G′, e, f, g will vanish and equation (13) will reduce to

$$V^2 = \frac{a\alpha^2 + b\beta^2 + c\gamma^2}{\rho^2}.$$

If the coordinate axes are so placed that

$$a > b > c,$$

the optic axes will lie in the X-Z plane, making equal angles ϕ with the axis of Z, which may be determined by the equation

$$\tan^2\phi = \frac{a-b}{b-c} = \frac{p^2(A-B) - 4\pi^2(A'-B')}{p^2(B-C) - 4\pi^2(B'-C')}.$$

To get a rough idea of the manner in which ϕ varies with the period, we may regard A, B, C, A′, B′, C′ as constant in this equation.

But since the lengths of the axes of the ellipsoid (a, b, etc.) vary with the period, it may easily happen that the order of the axes with respect to magnitude is not the same for all colors. In that case, the optic axes for certain colors will lie in one of the principal planes, and for other colors in another. For the color at which the change takes place, the two optic axes will coincide. The differential coefficient $\frac{d\phi}{dp}$ becomes infinitely great as the optic axes approach coincidence.

In crystals of the monoclinic system, each of the three ellipsoids will have an axis perpendicular to the plane of symmetry. We may choose this direction for the axis of X. Then F, G, F′, G′, f, g, will vanish and equation (13) will reduce to

$$V^2 = \frac{a\alpha^2 + b\beta^2 + c\gamma^2 + e\beta\gamma}{\rho^2}.$$

The angle θ made by one of the axes of the ellipsoid (a, b, etc.) in the plane of symmetry with the axis of Y and measured toward the axis of Z, is determined by the equation

$$\tan 2\theta = \frac{e}{c-b} = \frac{p^2E - 4\pi^2E'}{p^2(C-B) - 4\pi^2(C'-B')}.$$

To get a rough idea of the *dispersion* of the axes of the ellipsoid (a, b, etc.) in the plane of symmetry, we may regard B, C, E, B′, C′, E′, as constant in this equation, and suppose the axis of Y so placed as to make E vanish.

It is evident that in this system the plane of the optic axes will be fixed, or will rotate about one of the lines which bisect the angles made by the optic axes, according as the mean axis of the ellipsoid

(a, b, etc.) is perpendicular to the plane of symmetry or lies in that plane. In the first case the *dispersion* of the two optic axes will be unequal. The same crystal, however, with light of different colors, or at different temperatures, may afford an example of each case.

In crystals of the triclinic system, since the ellipsoids (A, B, etc.) and (A′, B′, etc.) are determined by considerations of a different nature, and there are no relations of symmetry to cause a coincidence in the directions of their axes, there will not in general be any such coincidence. Therefore the three axes of the ellipsoid (a, b, etc.), that is, the two lines which bisect the angles of the optic axes and their common normal, will vary in position with the color of the light.

16. It appears from this foregoing discussion that by the electromagnetic theory of light we may not only account for the dispersion of colors (including the dispersion of the lines which bisect the angles of the optic axes in doubly refracting media), but may also obtain Fresnel's laws of double refraction for every kind of homogeneous light *without neglect of the quantities which determine the dispersion of colors.*

But a closer approximation than that of this paper will be necessary to explain the phenomena of circularly polarizing media, which depend on very minute differences of wave-velocity, represented perhaps by a few units in the sixth significant figure of the index of refraction. That the degree of approximation which will give the laws of circular and elliptic polarization will not add any terms to the equations of this paper, except such as vanish for media which do not exhibit this phenomenon, will be shown in another number of this Journal.

XII.

ON DOUBLE REFRACTION IN PERFECTLY TRANSPARENT MEDIA WHICH EXHIBIT THE PHENOMENA OF CIRCULAR POLARIZATION.

[*American Journal of Science*, ser. 3, vol. XXIII, pp. 460-476, June, 1882.]

1. In the April number of this Journal,* the velocity of propagation of a system of plane waves of light, regarded as oscillating electrical fluxes, was discussed with such a degree of approximation as would account for the dispersion of colors and give Fresnel's laws of double refraction. It is the object of this paper to supplement that discussion by carrying the approximation so much further as is necessary in order to embrace the phenomena of circularly polarizing media.

2. If we imagine all the velocities in any progressive system of plane waves to be reversed at a given instant without affecting the displacements, and the system of wave-motion thus obtained to be superposed upon the original system, we obtain a system of stationary waves having the same wave-length and period of oscillation as the original progressive system. If we then reduce the magnitude of the displacements in the uniform ratio of two to one, they will be identical, at an instant of maximum displacement, with those of the original system at the same instant.

Following the same method as in the paper cited, let us especially consider the system of stationary waves, and divide the whole displacement into the *regular* part, represented by ξ, η, ζ, and the *irregular* part, represented by ξ', η', ζ', in accordance with the definitions of § 2 of that paper.

3. The *regular* part of the displacement is subject to the equations of wave-motion, which may be written (in the most general case of plane stationary waves)

$$\left.\begin{aligned} \xi &= \left(a_1 \cos 2\pi \frac{u}{l} + a_2 \sin 2\pi \frac{u}{l}\right) \cos 2\pi \frac{t}{p}, \\ \eta &= \left(\beta_1 \cos 2\pi \frac{u}{l} + \beta_2 \sin 2\pi \frac{u}{l}\right) \cos 2\pi \frac{t}{p}, \\ \zeta &= \left(\gamma_1 \cos 2\pi \frac{u}{l} + \gamma_2 \sin 2\pi \frac{u}{l}\right) \cos 2\pi \frac{t}{p}, \end{aligned}\right\} \quad (1)$$

* See page 182 of this volume.

where l denotes the wave-length, p the period of oscillation, u the distance of the point considered from the wave-plane passing through the origin, α_1, β_1, γ_1 the amplitudes of the displacements ξ, η, ζ in the wave-plane passing through the origin, and α_2, β_2, γ_2 their amplitudes in a wave-plane one-quarter of a wave-length distant and on the side toward which u increases. If we also write L, M, N for the direction-cosines of the wave-normal drawn in the direction in which u increases, we shall have the following necessary relations:

$$L^2+M^2+N^2=1, \tag{2}$$

$$u=Lx+My+Nz, \tag{3}$$

$$L\alpha_1+M\beta_1+N\gamma_1=0, \quad L\alpha_2+M\beta_2+N\gamma_2=0. \tag{4}$$

4. That the *irregular* part of the displacement (ξ', η', ζ') at any given point is a simple harmonic function of the time, having the same period and phase as the regular part of the displacement (ξ, η, ζ), may be proved by the single principle of superposition of motions, and is therefore to be regarded as exact in a discussion of this kind. But the further conclusion of the preceding paper (§ 4), "that the values of ξ', η', ζ' at any given point in the medium are capable of expression as linear functions of ξ, η, ζ in a manner which shall be independent of the time and of the orientation of the wave-planes and the distance of a nodal plane from the point considered, so long as the period of oscillation remains the same," is evidently only approximative, although a very close approximation. A very much closer approximation may be obtained, if we regard ξ', η', ζ', at any given point of the medium and for light of a given period, as linear functions of ξ, η, ζ and the nine differential coefficients

$$\frac{d\xi}{dx}, \quad \frac{d\eta}{dx}, \quad \frac{d\zeta}{dx}, \quad \frac{d\xi}{dy}, \quad \text{etc.}$$

We shall write ξ, η, ζ *and diff. coeff.* to denote these twelve quantities.

From this it follows immediately that with the same degree of approximation $\dot{\xi}'$, $\dot{\eta}'$, $\dot{\zeta}'$ may be regarded, for a given point of the medium and light of a given period, as linear functions of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ and the differential coefficients of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ with respect to the coordinates. For these twelve quantities we shall write $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ *and diff. coeff.*

5. Let us now proceed to equate the statical energy of the medium at an instant of no velocity with its kinetic energy at an instant of no displacement. It will be convenient to estimate each of these quantities for a unit of volume.

6. The statical energy of an infinitesimal element of volume may be represented by $\sigma\, dv$, where σ is a quadratic function of the components of displacement $\xi+\xi'$, $\eta+\eta'$, $\zeta+\zeta'$. Since for that element of volume ξ', η', ζ' may be regarded as linear functions of ξ, η, ζ *and diff. coeff.*,

we may regard σ as a quadratic function of ξ, η, ζ *and diff. coeff.*, or as a linear function of the seventy-eight squares and products of these quantities. But the seventy-eight coefficients by which this function is expressed will vary with the position of the element of volume with respect to the surrounding molecules.

In estimating the statical energy for any considerable space by the integral

$$\int \sigma \, dv,$$

it will be allowable to substitute for the seventy-eight coefficients contained implicitly in σ their average values throughout the medium. That is, if we write s for a quadratic function of ξ, η, ζ, *and diff. coeff.* in which the seventy-eight coefficients are the space-averages of those in σ, the statical energy of any considerable space may be estimated by the integral

$$\int s \, dv.$$

(This will appear most distinctly if we suppose the integration to be first effected for a thin slice of the medium bounded by two wave-planes.) The seventy-eight coefficients of this function s are determined solely by the nature of the medium and the period of oscillation.

We may divide s into three parts, of which the first ($s_,$) contains the squares and products of ξ, η, ζ, the second ($s_{,,}$) contains the products of ξ, η, ζ with the differential coefficients, and the third ($s_{,,,}$) contains the squares and products of the differential coefficients. It is evident that the average statical energy of the whole medium per unit of volume is the space-average of s, and that it will consist of three parts, which are the space-averages of $s_,$, $s_{,,}$, and $s_{,,,}$, respectively. These parts we may call $S_,$, $S_{,,}$, and $S_{,,,}$. Only the first of these was considered in the preceding paper.

Now the considerations which justify us in neglecting, for an approximate estimate, the terms of s which contain the differential coefficients of ξ, η, ζ with respect to the coordinates, will apply with especial force to the terms which contain the squares and products of these differential coefficients. Therefore, to carry the approximation one step beyond that of the preceding paper, it will only be necessary to take account of $s_,$ and $s_{,,}$ and of $S_,$ and $S_{,,}$.

7. We may set

$$s_, = A\xi^2 + B\eta^2 + C\zeta^2 + E\eta\zeta + F\zeta\xi + G\xi\eta, \tag{5}$$

where, for a given medium and light of a given period, A, B, C, E, F, G are constant.

Since the average values of

$$\sin^2 2\pi \frac{u}{l}, \quad \cos^2 2\pi \frac{u}{l}, \quad \sin 2\pi \frac{u}{l} \cos 2\pi \frac{u}{l}$$

are respectively $\frac{1}{2}$, $\frac{1}{2}$, and 0, and since at the time to be considered

$$\cos^2 2\pi \frac{t}{p} = 1,$$

it will appear from inspection of equations (1) that

$$S_{,} = \tfrac{1}{2}(A\alpha_1^2 + B\beta_1^2 + C\gamma_1^2 + E\beta_1\gamma_1 + F\gamma_1\alpha_1 + G\alpha_1\beta_1) + \tfrac{1}{2}(A\alpha_2^2 + B\beta_2^2 + C\gamma_2^2 + E\beta_2\gamma_2 + F\gamma_2\alpha_2 + G\alpha_2\beta_2). \quad (6)$$

This is the first part of the statical energy of the whole medium per unit of volume.

8. The second part of the statical energy of the whole medium per unit of volume ($S_{,,}$) is the space-average of $s_{,,}$, which is a linear function of the twenty-seven products of ξ, η, ζ with their differential coefficients with respect to the coordinates. Now since

$$\xi\frac{d\xi}{dx} = \tfrac{1}{2}\frac{d(\xi^2)}{dx}, \quad \eta\frac{d\eta}{dx} = \tfrac{1}{2}\frac{d(\eta^2)}{dx}, \text{ etc.},$$

the space-average of such products will be zero, and they will contribute nothing to the value of $S_{,,}$. There will be nine of these products, in which the same component of displacement appears twice. The remaining eighteen products may be divided into pairs according to the letters which they contain, as

$$\eta\frac{d\xi}{dx} \text{ and } \xi\frac{d\eta}{dx}.$$

A linear function of the eighteen products may also be regarded as a linear function of the sums and differences of the products in such pairs. But since

$$\eta\frac{d\xi}{dx} + \xi\frac{d\eta}{dx} = \frac{d(\eta\xi)}{dx},$$

the terms of $s_{,,}$ containing such sums will contribute nothing to the value of $S_{,,}$. We have left a linear function of the nine differences

$$\eta\frac{d\xi}{dx} - \xi\frac{d\eta}{dx}, \quad \zeta\frac{d\xi}{dx} - \xi\frac{d\zeta}{dx}, \quad \xi\frac{d\eta}{dx} - \eta\frac{d\xi}{dx}, \text{ etc.}$$

(the unwritten expressions being obtained by substituting in the denominators dy and dz for dx), which constitutes the part of $s_{,,}$ that we have to consider. $S_{,,}$ is therefore a linear function of the space-averages of these nine quantities. But by (3)

$$\eta\frac{d\xi}{dx} - \zeta\frac{d\eta}{dx} = L\left(\eta\frac{d\xi}{du} - \zeta\frac{d\eta}{du}\right),$$

and the space-average of this, at a moment of maximum displacement, is by (1)

$$\frac{2\pi L}{l}(\beta_1\gamma_2 - \gamma_1\beta_2).$$

By such reductions it appears that $lS_{''}$ is a linear function of the nine products of L, M, N with

$$\beta_1\gamma_2-\gamma_1\beta_2, \quad \gamma_1\alpha_2-\alpha_1\gamma_2, \quad \alpha_1\beta_2-\beta_1\alpha_2.$$

Now if we set

$$\Theta=L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2), \qquad (7)$$

we have by (4) and (2)

$$L\Theta=\beta_1\gamma_2-\gamma_1\beta_2, \quad M\Theta=\gamma_1\alpha_2-\alpha_1\gamma_2, \quad N\Theta=\alpha_1\beta_2-\beta_1\alpha_2. \qquad (8)$$

Therefore $lS_{''}$ is a linear function of the nine products of L, M, N with $L\Theta$, $M\Theta$, $N\Theta$. That is, $lS_{''}$ is the product of Θ and a quadratic function of L, M and N. We may therefore write

$$S_{''}=\frac{\Phi}{l}\Theta=\frac{\Phi}{l}[L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2)], \qquad (9)$$

where Φ is a quadratic function of L, M and N, dependent, however, on the nature of the medium and the period of oscillation.

9. It will be useful to consider more closely the geometrical significance of the quantity Θ. For this purpose it will be convenient to have a definite understanding with respect to the relative position of the coordinate axes.

We shall suppose that the axes of X, Y, and Z are related in the same way as lines drawn to the right, forward and upward, so that a rotation from X to Y appears clockwise to one looking in the direction of Z.

Now if from any same point, as the origin of coordinates, we lay off lines representing in direction and magnitude the displacements in all the different wave-planes, we obtain an ellipse, which we may call the *displacement-ellipse*.* Of this, one radius vector (ρ_1) will have the components α_1, β_1, γ_1, and another (ρ_2) the components α_2, β_2, γ_2. These will belong to conjugate diameters, each being parallel to the tangent at the extremity of the other. The area of the ellipse will therefore be equal to the parallelogram of which ρ_1 and ρ_2 are two sides, multiplied by π. Now it is evident that $\beta_1\gamma_2-\gamma_1\beta_2$, $\gamma_1\alpha_2-\alpha_1\gamma_2$, $\alpha_1\beta_2-\alpha_1\beta_2$ are numerically equal to the projections of this parallelogram on the planes of the coordinate axes, and are each positive or negative according as a revolution from ρ_1 to ρ_2 appears clockwise or counter-clockwise to one looking in the direction of the proper coordinate axis. Hence, Θ will be numerically equal to the parallelogram, that is, to the area of the displacement-ellipse divided by π, and will be positive or negative

* This ellipse, which represents the simultaneous displacements in different parts of the field, will also represent the successive displacements at any same point in the corresponding system of progressive waves.

according as a revolution from ρ_1 to ρ_2 appears clockwise or counter-clockwise to one looking in the direction of the wave-normal. Since ρ_1 and ρ_2 are determined by displacements in planes one-quarter of a wave-length distant from each other, and the plane to which the latter relates lies on the side toward which the wave-normal is drawn, it follows that Θ is positive or negative according as the combination of displacements has the character of a right-handed or a left-handed screw.

10. The kinetic energy of the medium, which is to be estimated for an instant of no displacement, may be shown as in § 7 of the former paper (page 185 of this volume) to consist of two parts, of which one relates to the regular flux $(\dot\xi, \dot\eta, \dot\zeta)$, and the other to the irregular flux $(\dot\xi', \dot\eta', \dot\zeta')$. The first, in the notation of that paper, is represented by

$$\tfrac{1}{2}\int(\dot\xi\,\mathrm{Pot}\,\dot\xi+\dot\eta\,\mathrm{Pot}\,\dot\eta+\dot\zeta\,\mathrm{Pot}\,\dot\zeta)dv,$$

which reduces to

$$\frac{l^2}{2\pi}\int(\dot\xi^2+\dot\eta^2+\dot\zeta^2)\,dv.$$

By substitution of the values given by equations (1), we obtain for the kinetic energy due to the regular flux in a unit of volume

$$\mathrm{T}=\frac{\pi l^2}{p^2}(\alpha_1{}^2+\beta_1{}^2+\gamma_1{}^2+\alpha_2{}^2+\beta_2{}^2+\gamma_2{}^2). \qquad (10)$$

11. The kinetic energy of the irregular part of the flux is represented by the volume-integral

$$\int\tfrac{1}{2}(\dot\xi'\,\mathrm{Pot}\,\dot\xi'+\dot\eta'\,\mathrm{Pot}\,\dot\eta'+\dot\zeta'\,\mathrm{Pot}\,\dot\zeta')\,dv.$$

Now, since $\dot\xi'$ $\dot\eta'$, $\dot\zeta'$ are everywhere linear functions of $\dot\xi$, $\dot\eta$, $\dot\zeta$ *and diff. coeff.* (see § 4), and since the integrations implied in the notation *Pot* may be confined to a sphere of which the radius is small in comparison with a wave length,* and since within such a sphere $\dot\xi$, $\dot\eta$, $\dot\zeta$ *and diff. coeff.* are sufficiently determined (in a linear form), by the values of the same twelve quantities at the center of the sphere, it follows that $\mathrm{Pot}\,\dot\xi'$, $\mathrm{Pot}\,\dot\eta'$, $\mathrm{Pot}\,\dot\zeta'$ must be linear functions of the values of $\dot\xi$, η, $\dot\zeta$ *and diff. coeff.* at the point for which the potential is sought. Hence,

$$\tfrac{1}{2}(\dot\xi'\,\mathrm{Pot}\,\dot\xi'+\dot\eta'\,\mathrm{Pot}\,\dot\eta'+\dot\zeta'\,\mathrm{Pot}\,\zeta')$$

will be a quadratic function of $\dot\xi$, $\dot\eta$, $\dot\zeta$ *and diff. coeff.* But the seventy-eight coefficients by which this function is expressed will vary with the position of the point considered with respect to the surrounding molecules.

* See § 9 of the former paper, on page 187 of this volume.

Yet, as in the case of the statical energy, we may substitute the average values of these coefficients for the coefficients themselves in the integral by which we obtain the energy of any considerable space. The kinetic energy due to the irregular part of the flux is thus reduced to a quadratic function of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ *and diff. coeff.* which has constant coefficients for a given medium and light of a given period.

The function may be divided into three parts, of which the first contains the squares and products of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$, the second the products of $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ with their differential coefficients, and the third, which may be neglected, the squares and products of the differential coefficients.

We may proceed with the reduction precisely as in the case of the statical energy, except that the differentiations with respect to the time will introduce the constant factor $\frac{4\pi^2}{p^2}$. This will give for the first part of the kinetic energy of the irregular flux per unit of volume

$$\begin{aligned} T'_{,} = {} & \frac{2\pi^2}{p^2}(A'\alpha_1^2 + B'\beta_1^2 + C'\gamma_1^2 + E'\beta_1\gamma_1 + F'\gamma_1\alpha_1 + G'\alpha_1\beta_1) \\ & + \frac{2\pi^2}{p^2}(A'\alpha_2^2 + B'\beta_2^2 + C'\gamma_2^2 + E'\beta_2\gamma_2 + F'\gamma_2\alpha_2 + G'\alpha_2\beta_2), \quad (11) \end{aligned}$$

and for the second part of the same

$$\begin{aligned} T'_{,,} = {} & \frac{4\pi^2\Phi'}{p^2 l}\Theta \\ = {} & \frac{4\pi^2\Phi'}{p^2 l}[L(\beta_1\gamma_2 - \gamma_1\beta_2) + M(\gamma_1\alpha_2 - \alpha_1\gamma_2) + N(\alpha_1\beta_2 - \beta_1\alpha_2)], \quad (12) \end{aligned}$$

where A′, B′, C′, E′, F′, G′ are constant, and Φ' a quadratic function of L, M, and N, for a given medium and light of a given period.

12. Equating the statical and kinetic energies, we have

$$S_{,} + S_{,,} = T + T'_{,} + T'_{,,},$$

that is, by equations (6), (9), (10), (11), and (12),

$$\begin{aligned} & \tfrac{1}{2}(A\alpha_1^2 + B\beta_1^2 + C\gamma_1^2 + E\beta_1\gamma_1 + F\gamma_1\alpha_1 + G\alpha_1\beta_1) \\ & + \tfrac{1}{2}(A\alpha_2^2 + B\beta_2^2 + C\gamma_2^2 + E\beta_2\gamma_2 + F\gamma_2\alpha_2 + G\alpha_2\beta_2) \\ & + \frac{\Phi}{l}[L(\beta_1\gamma_2 - \gamma_1\beta_2) + M(\gamma_1\alpha_2 - \alpha_1\gamma_2) + N(\alpha_1\beta_2 - \beta_1\alpha_2)] \\ & = \frac{\pi l^2}{p^2}(\alpha_1^2 + \beta_1^2 + \gamma_1^2 + \alpha_2^2 + \beta_2^2 + \gamma_2^2) \\ & + \frac{2\pi^2}{p^2}(A'\alpha_1^2 + B'\beta_1^2 + C'\gamma_1^2 + E'\beta_1\gamma_1 + F'\gamma_1\alpha_1 + G'\alpha_1\beta_1) \\ & + \frac{2\pi^2}{p^2}(A'\alpha_2^2 + B'\beta_2^2 + C'\gamma_2^2 + E'\beta_2\gamma_2 + F'\gamma_2\alpha_2 + G'\alpha_2\beta_2) \\ & + \frac{4\pi^2\Phi'}{p^2 l}[L(\beta_1\gamma_2 - \gamma_1\beta_2) + M(\gamma_1\alpha_2 - \alpha_1\gamma_2) + N(\alpha_1\beta_2 - \beta_1\alpha_2)]. \quad (13) \end{aligned}$$

If we set
$$a=\frac{A}{2\pi}-\frac{2\pi A'}{p^2},\quad b=\frac{B}{2\pi}-\frac{2\pi B'}{p^2},\quad \text{etc.,} \tag{14}$$

and
$$\phi=\frac{\Phi}{2\pi p}-\frac{2\pi\Phi'}{p^3}, \tag{15}$$

the equation reduces to

$$\begin{aligned}&a\alpha_1^2+b\beta_1^2+c\gamma_1^2+e\beta_1\gamma_1+f\gamma_1\alpha_1+g\alpha_1\beta_1\\&+a\alpha_2^2+b\beta_2^2+c\gamma_2^2+e\beta_2\gamma_2+f\gamma_2\alpha_2+g\alpha_2\beta_2\\&+\frac{2p\phi}{l}[L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2)]\\&=\frac{l^2}{p^2}(\alpha_1^2+\beta_1^2+\gamma_1^2+\alpha_2^2+\beta_2^2+\gamma_2^2),\end{aligned} \tag{16}$$

where a, b, c, e, f, g are constant, and ϕ a quadratic function of L, M, N, for a given medium and light of a given period.

13. Now this equation, which expresses a relation between the constants of the equations of wave-motion (1), will apply, with those equations, not only to such vibrations as actually take place, but also to such as we may imagine to take place under the influence of constraints determining the type of vibration. The free or unconstrained vibrations, with which alone we are concerned, are characterized by this, that infinitesimal variations (by constraint) of the type of vibration, that is, of the ratios of the quantities α_1, β_1, γ_1, α_2, β_2, γ_2, will not affect the period by any quantity of the same order of magnitude.* These variations must however be consistent with equations (4), which require that

$$L\,d\alpha_1+M\,d\beta_1+N\,d\gamma_1=0,\quad L\,d\alpha_2+M\,d\beta_2+N\,d\gamma_2=0. \tag{17}$$

Hence, to obtain the conditions which characterize free vibration, we may differentiate equation (16) with respect to α_1, β_1, γ_1, α_2, β_2, γ_2, regarding all other letters as constant, and give to $d\alpha_1$, $d\beta_1$, $d\gamma_1$, $d\alpha_2$, $d\beta_2$, $d\gamma_2$, such values as are consistent with equations (17). Now $d\alpha_1$, $d\beta_1$, $d\gamma_1$, are independent of $d\alpha_2$, $d\beta_2$, $d\gamma_2$, and for either three variations, values proportional either to α_1, β_1, γ_1, or to α_2, β_2, γ_2, are possible. If, then, we differentiate equation (16) with respect to α_1, β_1, γ_1, and substitute first α_1, β_1, γ_1, and then α_2, β_2, γ_2, for $d\alpha_1$, $d\beta_1$, $d\gamma_1$, and also differentiate with respect to α_2, β_2, γ_2, with similar substitutions, we shall obtain all the independent equations which this principle will yield.

If we differentiate with respect to α_1, β_1, γ_1, and write α_1, β_1, γ_1 for $d\alpha_1$, $d\beta_1$, $d\gamma_1$, we obtain

$$\begin{aligned}&a\alpha_1^2+b\beta_1^2+c\gamma_1^2+e\beta_1\gamma_1+f\gamma_1\alpha_1+g\alpha_1\beta_1\\&+\frac{p\phi}{l}[L(\beta_1\gamma_2-\gamma_1\beta_2)+M(\gamma_1\alpha_2-\alpha_1\gamma_2)+N(\alpha_1\beta_2-\beta_1\alpha_2)]\\&=\frac{l^2}{p^2}(\alpha_1^2+\beta_1^2+\gamma_1^2).\end{aligned} \tag{18}$$

* Compare § 11 of the former paper, page 189 of this volume.

If we differentiate with respect to α_1, β_1, γ_1, and write α_2, β_2, γ_2 for $d\alpha_1$, $d\beta_1$, $d\gamma_1$, we obtain

$$2a\alpha_1\alpha_2+2b\beta_1\beta_2+2c\gamma_1\gamma_2+e(\beta_1\gamma_2+\gamma_1\beta_2)+f(\gamma_1\alpha_2+\alpha_1\gamma_2)$$
$$+g(\alpha_1\beta_2+\beta_1\alpha_2)=\frac{2l^2}{p^2}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2). \qquad (19)$$

If we differentiate with respect to α_2, β_2, γ_2, and write α_2, β_2, γ_2 for $d\alpha_2$, $d\beta_2$, $d\gamma_2$, we obtain

$$a\alpha_2^2+b\beta_2^2+c\gamma_2^2+e\beta_2\gamma_2+f\gamma_2\alpha_2+g\alpha_2\beta_2$$
$$+\frac{p\phi}{l}[\mathrm{L}(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}(\alpha_1\beta_2-\beta_1\alpha_2)]$$
$$=\frac{l^2}{p^2}(\alpha_2^2+\beta_2^2+\gamma_2^2). \qquad (20)$$

The equation derived by differentiating with respect to α_2, β_2, γ_2, and writing α_1, β_1, γ_1 for $d\alpha_2$, $d\beta_2$, $d\gamma_2$, is identical with (19). We should also observe that equations (18) and (20) by addition give equation (16), which therefore will not need to be considered in addition to the last three equations.

14. The geometrical signification of our equations may now be simplified by a suitable choice of the position of the origin of coordinates, which is as yet wholly arbitrary.

We shall hereafter suppose that the origin is placed in a plane of maximum or minimum displacement,* if such there are. In the case of circular polarization, in which the displacements are everywhere equal, its position is immaterial. The lines ρ_1 and ρ_2, of which α_1, β_1, γ_1 and α_2, β_2, γ_2 are respectively the components, will now be the semi-axes of the displacement-ellipse, and therefore at right angles. (See § 9.) The case of circular polarization will not constitute any exception. Hence,

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0, \qquad (21)$$

and by § 9,

$$\Theta=\mathrm{L}(\beta_1\gamma_2-\gamma_1\beta_2)+\mathrm{M}(\gamma_1\alpha_2-\alpha_1\gamma_2)+\mathrm{N}(\alpha_1\beta_2-\beta_1\alpha_2)=\pm\rho_1\rho_2, \qquad (22)$$

where we are to read + or − in the last member according as the system of displacements has the character of a right-handed or a left-handed screw.

15. Equation (19) is now reduced to the form

$$2a\alpha_1\alpha_2+2b\beta_1\beta_2+2c\gamma_1\gamma_2+e(\beta_1\gamma_2+\gamma_1\beta_2)$$
$$+f(\gamma_1\alpha_2+\alpha_1\gamma_2)+g(\alpha_1\beta_2+\beta_1\alpha_2)=0, \qquad (23)$$

* The reader will perceive that an earlier limitation of the position of the origin by a supposition of this nature, involving a limitation of the values of α_1, β_1, γ_1, α_2, β_2, γ_2, would have been embarrassing in the operations of the last paragraph.

which has a very simple geometrical signification. If we consider the ellipsoid

$$ax^2+by^2+cz^2+eyz+fzx+gxy, \tag{24}$$

and especially its central section by a plane parallel to the planes of the wave-system which we are considering, it will easily appear that the equation

$$2ax_1x_2+2by_1y_2+2cz_1z_2+e(y_1z_2+z_1y_2)$$
$$+f(z_1x_2+x_1z_2)+g(x_1y_2+y_1x_2)=0$$

will hold of any two points x_1, y_1, z_1 and x_2, y_2, z_2 which belong to conjugate diameters of this central section. Therefore equation (23) expresses that the displacements α_1, β_1, γ_1 and α_2, β_2, γ_2 are parallel to conjugate diameters of the central section of the ellipsoid (24) by a wave-plane. But since the displacements α_1, β_1, γ_1 and α_2, β_2, γ_2 are also at right angles to each other, it follows that they are parallel to the axes of the central section of the ellipsoid (24) by a wave-plane. That is:—The axes of the displacement-ellipse coincide in direction with those of a central section of the ellipsoid (24) by a wave-plane.

16. If we write U_1, U_2 for the reciprocals of the semi-axes of the central section of the ellipsoid (24) by a wave-plane, U_1 being the reciprocal of the one to which the displacement α_1, β_1, γ_1 is parallel, we have

$$a\alpha_1^2+b\beta_1^2+c\gamma_1^2+e\beta_1\gamma_1+f\gamma_1\alpha_1+g\alpha_1\beta_1=U_1^2(\alpha_1^2+\beta_1^2+\gamma_1^2), \tag{25}$$

as is at once evident if we substitute the coordinates of an extremity of the axis for the proportional quantities α_1, β_1, γ_1. So also

$$a\alpha_2^2+b\beta_2^2+c\gamma_2^2+e\beta_2\gamma_2+f\gamma_2\alpha_2+g\alpha_2\beta_2=U_2^2(\alpha_2^2+\beta_2^2+\gamma_2^2). \tag{26}$$

If we write V for the velocity of propagation of the system of progressive waves corresponding to the system of stationary waves which we have been considering, we shall have

$$V=\frac{l}{p}. \tag{27}$$

By equations (22), (25), and (26), equations (18) and (20) are reduced to the form

$$U_1^2\rho_1^2\pm\frac{\phi}{V}\rho_1\rho_2=V^2\rho_1^2, \qquad U_2^2\rho_2^2\pm\frac{\phi}{V}\rho_1\rho_2=V^2\rho_2^2, \tag{28}$$

where we are to read + or − according as the disturbance has the character of a right-handed or a left-handed screw. In a progressive system of waves, when the combination of displacements has the character of a right-handed screw, the rotations will be such as appear clockwise to the observer, who looks in the direction opposite to that of the propagation of light. We shall call such a ray *right-handed.*

We may here observe that in case $\phi=0$ the solution of these

equations is very simple. We have necessarily either $\rho_2=0$ and $V^2=U_1^2$, or $\rho_1=0$ and $V^2=U_2^2$. In this case, the light is linearly polarized, and the directions of oscillation and the velocities of propagation are given by Fresnel's law. Experiment has shown that this is the usual case. We wish, however, to investigate the case in which ϕ does not vanish. Since the term containing ϕ arises from the consideration of those quantities which it was allowable to neglect in the first approximation, we may assume that ϕ is always very small in comparison with V^3, U_1^3, or U_2^3.

17. Equations (28) may be written

$$V^2-U_1^2=\pm\frac{\phi}{V}\frac{\rho_2}{\rho_1},\quad V^2-U_2^2=\pm\frac{\phi}{V}\frac{\rho_1}{\rho_2}. \tag{29}$$

By multiplication we obtain

$$V^2(V^2-U_1^2)(V^2-U_2^2)=\phi^2. \tag{30}$$

Since ϕ is a very small quantity, it is evident from inspection of this equation that it will admit three values of V^2, of which one will be a very little greater than the greater of the two quantities U_1^2 and U_2^2, another will be a very little less than the less of the same two quantities, and the third will be a very small quantity. It is evident that the values of V^2 with which we have to do are those which differ but little from U_1^2 and U_2^2.*

For the numerical computation of V, when U_1, U_2, and ϕ are known numerically, we may divide the equation by V^2, and then solve it as if the second member were known. This will give

$$V^2=\frac{U_1^2+U_2^2}{2}\pm\sqrt{\frac{\phi^2}{V^2}+\frac{(U_1^2-U_2^2)^2}{4}}. \tag{31}$$

By substituting U_1U_2 for V^2 in the second member, we may obtain a close approximation to the two values of V^2. Each of the values obtained may be improved by substitution of that value for V^2 in the second member of the equation.

For either value of V^2, we may easily find the ratio of ρ_1 to ρ_2, that is, the ratio of the axes of the displacement-ellipse, from one of equations (29), or from the equation

$$\frac{\rho_2^2}{\rho_1^2}=\frac{V^2-U_1^2}{V^2-U_2^2} \tag{32}$$

obtained by combining the two.

* We should not attribute any physical significance to the third value of V^2. For this value would imply a wave-length very small in comparison with the length of ordinary waves of light, and with respect to which our fundamental assumption that the wave-length is very great in comparison with the distances of contiguous molecules would be entirely false. Our analysis, therefore, furnishes no reason for supposing that any such velocities are possible for the propagation of electrical disturbances.

In equations (29), we are to read + or − in the second members, according as the ray is right-handed or left-handed. (See § 16.) It follows that if the value of ϕ is positive, the greater velocity will belong to a right-handed ray, and the smaller to a left-handed, but if the value of ϕ is negative, the opposite is the case. Except when $\phi = 0$, and the polarization is linear, there will be one right-handed and one left-handed ray for any given wave-normal and period.

18. When $U_1 = U_2$, equations (29) give

$$\rho_1 = \rho_2, \quad V^2 = U^2 \pm \frac{\phi}{V},$$

where U represents the common value of U_1 and U_2. The polarization is therefore circular. The converse is also evident from equations (29), viz., that a ray can be circularly polarized only when the direction of its wave-normal is such that $U_1 = U_2$. Such a direction, which is determined by a circular section of the ellipsoid (24) precisely as an optic axis of a crystal which conforms to Fresnel's law of double refraction, may be called an optic axis, although its physical properties are not the same as in the more ordinary case.* If we write V_R and V_L, respectively, for the wave-velocities of the right-handed and left-handed rays, we have

$$V_R^2 = U^2 + \frac{\phi}{V_R}, \quad V_L^2 = U^2 - \frac{\phi}{V_L}; \tag{33}$$

whence

$$V_R^2 - V_L^2 = \phi\left(\frac{1}{V_R} + \frac{1}{V_L}\right) = \phi\,\frac{V_R + V_L}{V_R V_L},$$

and

$$V_R - V_L = \frac{\phi}{V_R V_L}. \tag{34}$$

The phenomenon best observed with respect to an optic axis is the rotation of the plane of linearly polarized light. If we denote by θ the amount of this rotation per unit of the distance traversed by the wave-plane, regarding it as positive when it appears clockwise to the

* Our experimental knowledge of circularly or elliptically polarizing media is confined to such as are optically either isotropic or uniaxial. The general theory of such media, embracing the case of two optic axes, has however been discussed by Professor von Lang ("Theorie der Circularpolarization," *Sitz.-Ber. Wiener Akad.*, vol. lxxv, p. 719). The general results of the present paper, although derived from physical hypotheses of an entirely different nature, are quite similar to those of the memoir cited. They would become identical, the writer believes, by the substitution of a constant for $\frac{p\phi}{l}$ or $\frac{\phi}{V}$ in the equations of this paper. (See especially equations (18), (20), (28).)

That a complete discussion of the subject on any theory must include the case of biaxial media having the property of circular or elliptical polarization, is evident from the consideration that it must at least be possible to produce examples of such media artificially. An isotropic or uniaxial crystal may be made biaxial by pressure. If it has the property of circular and elliptic polarization, that property cannot be wholly destroyed by the application of small pressures.

observer, who looks in the direction opposite to that of the propagation of the light,* we have

$$\theta=\frac{\pi}{p}\left(\frac{1}{V_L}-\frac{1}{V_R}\right). \tag{35}$$

By the preceding equation, this reduces to

$$\theta=\frac{\pi\phi}{pV_R^2V_L^2}. \tag{36}$$

Without any appreciable error, we may substitute U^4 for $V_R^2V_L^2$, which will give†

$$\theta=\frac{\pi\phi}{pU^4}. \tag{37}$$

19. Since these equations involve unknown functions of the period they will not serve for an exact determination of the relation between θ and the period. For a rough approximation, however, we may assume that the manner in which the general displacement in any small part of the medium distributes itself among the molecules and intermolecular spaces is independent of the period, being determined entirely by the values of ξ, η, ζ, and their differential coefficients with respect to the coordinates.‡ For a fixed direction of the wave-normal, Φ and Φ' will then be constant. Now equations (15) and (36) give

$$\theta=\frac{\Phi}{2p^2V_R^2V_L^2}-\frac{2\pi^2\Phi'}{p^4V_R^2V_L^2}. \tag{38}$$

To express this result in terms of the quantities directly observed, we may use the equations

$$p=\frac{\lambda}{k},\quad V_R=\frac{k}{n_R},\quad V_L=\frac{k}{n_L},\quad U=\frac{k}{n},$$

where k denotes the velocity of light *in vacuo*, λ the wave-length *in vacuo* of the light employed, n_R, n_L the absolute indices of refraction of the two rays, and n the index for the optic axis as derived from the ellipsoid (24) by Fresnel's law. We thus obtain

$$\theta=\frac{\Phi n_R^2n_L^2}{2k^2\lambda^2}-\frac{2\pi^2\Phi'n_R^2n_L^2}{\lambda^4}. \tag{39}$$

* When the rotation of the plane of polarization appears clockwise to the observer, it has the character of a *left-handed* screw. But the circularly polarized ray to which V_R relates, the rotation of which also appears clockwise to the observer, has the character of a *right-handed* screw.

† The degree of accuracy of this substitution may be shown as follows. By (33)

$$V_R(V_R^2-U^2)=V_L(U^2-V_L^2),$$

whence

$$V_R^2+V_RV_L=(V_R+V_L)U^2,$$
$$V_R^2-V_RV_L+V_L^2=U^2,$$
$$V_RV_L=U^2-(V_R-V_L)^2.$$

‡ Compare § 12 of the former paper, on page 189 of this volume.

In the case of uniaxial crystals, the direction of the optic axis is fixed. We may therefore write

$$\theta = n_R^2 n_L^2 \left(\frac{K}{\lambda^2} + \frac{K'}{\lambda^4}\right), \tag{40}$$

regarding K and K′ as constants. If we had used equation (37), we should have had the factor n^4 instead of $n_R^2 n_L^2$. Since this factor varies but slowly with λ, it may be neglected, if its omission is compensated in the values of K and K′. The formula being only approximative, such a simplification will not necessarily render it less accurate.

20. But without any such assumption as that contained in the last paragraph, we may easily obtain formulæ for the experimental determination of Φ and Φ' for the optic axis of a uniaxial crystal. Considerations analogous to those of § 13 of the former paper (page 190 of this volume), show that in differentiating equation (39) we may regard Φ and Φ' as constant, although they may actually vary with λ. This equation may be written

$$\frac{\theta\lambda^2}{n^4} = \frac{\Phi}{2k^2} - \frac{2\pi^2\Phi'}{\lambda^2}. \tag{41}$$

Therefore,

$$\frac{d\left(\frac{\theta\lambda^2}{n^4}\right)}{d\left(\frac{1}{\lambda^2}\right)} = -2\pi^2\Phi'. \tag{42}$$

When Φ' has been determined by this equation, Φ may be found from the preceding.

21. If we wish to represent ϕ geometrically, like U_1 and U_2, we may construct the surfaces

$$Ax^2 + By^2 + Cz^2 + Eyz + Fzx + Gxy = \pm 1, \tag{43}$$

the coefficients A, B, etc., being the same by which ϕ is expressed in terms of L^2, M^2, etc. The numerical value of ϕ, for any direction of the wave-normal, will thus be represented by the square of the reciprocal of the radius vector of the surface drawn in the same direction. The positive or negative character of ϕ must be separately indicated. There are here two cases to be distinguished. If the sign of ϕ is the same in all directions, the surface will be an ellipsoid, and we have only to know whether all the values of ϕ are to be taken positively or all negatively. But if ϕ is positive for some directions and negative for others, the surface will consist of two conjugate hyperboloids, to one of which the positive, and to the other the negative values belong.

22. The manner in which the ellipsoid (24) may be partially determined by the relations of symmetry which the medium may possess, has been sufficiently discussed in the former paper.

With respect to the quantity ϕ, and the surfaces which determine it, the following principle is of fundamental importance. If one body is identical in its internal structure with the image by reflection of another, the values of ϕ in corresponding lines in the two bodies will be numerically equal but have opposite signs.*

It follows that if a body is identical in internal structure with its own image by reflection, the value of ϕ (if not zero for all directions) must be positive for some directions and negative for others. Moreover, the above described surface by which ϕ is represented must consist of two conjugate hyperboloids, of which one is identical in form with the image by reflection of the other. This requires that the hyperboloids shall be right cylinders with conjugate rectangular hyperbolas for bases. A crystal characterized by such properties will belong to the tetragonal system. Since $\phi = 0$ for the optic axis, it would be difficult to distinguish a case of this kind from an ordinary uniaxial crystal, unless the ellipsoid (24) should approach very closely to a sphere.†

It is only in the very limited case described in the last paragraph that a medium which is identical in its internal structure with its image by reflection can have the property of circular or elliptic polarization. To media which are unlike their images by reflection, and have the property of circular polarization, we may apply the following general principles.

If the medium has any axis of symmetry, the ellipsoid or hyperboloids which represent the values of ϕ will have an axis in the same direction. If the medium after a revolution of less than 180° about any axis is equivalent to the medium in its first position, the ellipsoid or hyperboloids will have an axis of revolution in that direction.

23. The laws of the propagation of light in plane waves, which

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* The necessity of the opposite signs will perhaps appear most readily from the consideration that the direction of rotation of the plane of polarization must be opposite in the two bodies.

† There is no difficulty in conceiving of the constitution of a body which would have the properties described above. Thus, we may imagine a body with molecules of a spiral form, of which one-half are right-handed and one-half left-handed, and we may suppose that the motion of electricity is opposed by a less resistance within them than without. If the axes of the right-handed molecules are parallel to the axis of X, and those of the left-handed molecules to the axis of Y, their effects would counterbalance one another when the wave-normal is parallel to the axis of Z. But when the wave-normal (of a beam of linearly polarized light) is parallel to the axis of X, the left-handed molecules would produce a left-handed (negative) rotation of the plane of polarization, the right-handed molecules having no effect; and when the wave-normal is parallel to the axis of Y, the reverse would be the case.

have thus been derived from the single hypothesis that the disturbance by which light is transmitted consists of solenoidal electrical fluxes, and which apply to light of different colors and to the most general case of perfectly transparent and sensibly homogeneous media not subject to magnetic action,* are essentially those which are generally received as embodying the results of experiment. In no particular, so far as the writer is aware, do they conflict with the results of experiment, or require the aid of auxiliary and forced hypotheses to bring them into harmony therewith.

In this respect, the electromagnetic theory of light stands in marked contrast with that theory in which the properties of an elastic solid are attributed to the ether,—a contrast which was very distinct in Maxwell's derivation of Fresnel's laws from electrical principles, but becomes more striking as we follow the subject farther into its details, and take account of the want of absolute homogeneity in the medium, so as to embrace the phenomena of the dispersion of colors and circular and elliptical polarization.

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* The rotation of the plane of polarization which is produced by magnetic action has been discussed by Maxwell (*Treatise on Electricity and Magnetism*, vol. ii, chap. xxi), and by Rowland (*Amer. Journ. Math.*, vol. iii, p. 107).

XIII.

ON THE GENERAL EQUATIONS OF MONOCHROMATIC LIGHT IN MEDIA OF EVERY DEGREE OF TRANSPARENCY.

[*American Journal of Science*, ser 3, vol. XXV, pp. 107-118, February, 1883.]

1. THE last April and June numbers of this Journal* contain an investigation of the velocity of plane waves of light, in which they are regarded as consisting of solenoidal electrical fluxes in an indefinitely extended medium of uniform and very fine-grained structure. It was also supposed that the medium was perfectly transparent, although without discussion of the physical properties on which transparency depends, and that the electrical motions were not complicated by any distinctively magnetic phenomena.

In the present paper† the subject will be treated with more generality, so as to obtain the general equations of monochromatic light for media of every degree of transparency, whether sensibly homogeneous or otherwise, which have a very fine-grained molecular structure as measured by a wave-length of light. There will be no restriction with respect to magnetic influence, except that an oscillating magnetization of the medium will be excluded.‡

In order to conform as much as possible to the ordinary view of

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* See pages 182–194 and 195–210 of this volume.

† This paper contains, with some additional developments, the substance of a communication to the National Academy of Sciences in November, 1882.

‡ Where a body capable of magnetization is subjected to the influence of light (as when light is reflected from the surface of iron), there are two simple hypotheses which present themselves with respect to the magnetic state of the body. One is that the magnetic forces due to the light are not of sufficient duration to allow the molecular changes which constitute magnetization to take place to any sensible extent. The other is that the magnetization has a constant ratio to the magnetic force without regard to its duration. We might easily make a more general hypothesis which would embrace both of those mentioned as extreme cases, and which would be irreproachable from a theoretical stand-point; but it would complicate our equations to a degree which would not be compensated by their greater generality, since no phenomena depending on such magnetization have been observed, so far as the writer is aware, or are likely to be, except in a very limited class of cases.

For the purposes of this paper, therefore, it has seemed better to exclude media capable of magnetization, except so far as the first mentioned hypothesis may be applicable. But it does not appear that this requires us to exclude cases in which the medium is subject to the influence of a permanent magnetic force, such as produces the phenomenon of the magnetic rotation of the plane of polarization.

electrical phenomena,* we shall not introduce at first the hypothesis of Maxwell that electrical fluxes are solenoidal.† Our results, however, will be such as to require us to admit the substantial truth of this hypothesis, if we regard the processes involved in the transmission of light as electrical.

With regard to the undetermined questions of electrodynamic induction, we shall adopt provisionally that hypothesis which appears the most simple, yet proceed in such a manner that it will be evident exactly how our results must be altered, if we prefer any other hypothesis.

Electrical quantities will be treated as measured in electromagnetic units.

2. We must distinguish, as before, between the *actual* electrical displacements, which are too complicated to follow in detail with analysis, and which in their minutiæ elude experimental demonstration, and the displacements as *averaged* for spaces which are large enough to smooth out their minor irregularities, but not so large as to obliterate to any sensible extent those more regular features of the electrical motion, which form the subject of optical experiment. These spaces must therefore be large as measured by the least distances between molecules, but small as measured by a wave-length of light. We shall also have occasion to consider similar averages for other quantities, as electromotive force, the electrostatic potential, etc. It will be convenient to suppose that the space for which the average is taken is the same in all parts of the field,‡ say a sphere of uniform radius having its center at the point considered.

Whatever may be the quantities considered, such averages will be represented by the notation

$$[\quad]_{\text{Ave.}}$$

* It has, perhaps, retarded the acceptance of the electromagnetic theory of light that it was presented in connection with a theory of electrical action, which is probably more difficult to prove or disprove, and certainly presents more difficulties of comprehension, than the connection of optical and electrical phenomena, and which, as resting largely on *a priori* considerations, must naturally appear very differently to different minds. Moreover, the mathematical method by which the subject was treated, while it will remain a striking monument of its author's originality of thought, and profoundly modify the development of mathematical physics, must nevertheless, by its wide departure from ordinary methods, have tended to repel such as might not make it a matter of serious study.

† A flux is said to be *solenoidal* when it satisfies the conditions which characterize the motion of an incompressible fluid,—in other words, if u, v, w are the rectangular components of the flux, when

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

and the normal component of the flux is the same on both sides of any surfaces of discontinuity which may exist.

‡ This is rather to fix our ideas, than on account of any mathematical necessity. For the space for which the average is taken may in general be considerably varied without sensibly affecting the value of the average.

If, then, ξ, η, ζ denote the components of the actual displacement at the point considered,

$$[\xi]_{\text{Ave}}, \qquad [\eta]_{\text{Ave}}, \qquad [\zeta]_{\text{Ave}}$$

will represent the average values of these components in the small sphere about that point. These average values we shall treat as functions of the coordinates of the center of the sphere and of the time, and may call them, for brevity, the *average values* of ξ, η, ζ. But however they may be designated, it is essential to remember that it is a space-average for a certain very small space, and never a time-average, that is intended.

The object of this paper will be accomplished when we have expressed (explicitly or implicitly) the relations which subsist between the values of $[\xi]_{\text{Ave}}$, $[\eta]_{\text{Ave}}$, $[\zeta]_{\text{Ave}}$, at different times and in different parts of the field,—in other words, when we have found the conditions which these quantities must satisfy as functions of the time and the coordinates.

3. Let us suppose that luminous vibrations of any one period * are somewhere excited, and that the disturbance is propagated through the medium. The motions which are excited in any part of the medium, and the forces by which they are kept up, will be expressed by harmonic functions of the time, having the same period,† as may be proved by the single principle of the superposition of motions quite independently of any theory of the constitution of the medium, or of the nature of the motions, as electrical or otherwise. This is equally true of the actual motions, and of the averages which we are to consider. We may therefore set

$$\left.\begin{aligned} [\xi]_{\text{Ave}} &= a_1 \cos\frac{2\pi}{p}t + a_2 \sin\frac{2\pi}{p}t, \\ &\text{etc.,} \end{aligned}\right\} \qquad (1)$$

* There is no real loss of generality in making the light monochromatic, since in every case it may be divided into parts, which are separately propagated, and each of which is monochromatic to any required degree of approximation.

† It is of course possible that the expressions for the forces and displacements should have constant terms. But these will disappear, if the displacements are measured from the state of equilibrium about which the system vibrates, and we leave out of account in measuring the forces (and the electrostatic potential) that which would belong to the system in the state of equilibrium. To prevent misapprehension, it should be added that the term *electrical displacement* is *not* used in the restricted sense of *dielectric displacement* or *polarization*. The variation of the electrical displacement, as the term is used in this paper, constitutes what Maxwell calls the total motion of electricity or true current, and what he divides into two parts, which he distinguishes as the current of conduction and the variation of the electrical displacement. Such a division of the total motion of electricity is not necessary for the purposes of this paper, and the term displacement is used with reference to the total motion of electricity in a manner entirely analogous to that in which the term is ordinarily used in the theory of wave-motion.

where t denotes the time, p the period, and a_1, a_2, functions of the coordinates. It follows that

$$\left.\begin{aligned}[\ddot{\xi}]_{\text{Ave}} &= -\frac{4\pi^2}{p^2}[\xi]_{\text{Ave}}, \\ &\text{etc.}\end{aligned}\right\} \tag{2}$$

4. Now, on the electrical theory, these motions are excited by electrical forces, which are of two kinds, distinguished as electrostatic and electrodynamic. The electrostatic force is determined by the electrostatic potential. If we write q for the actual value of the potential, and $[q]_{\text{Ave}}$ for its value as averaged in the manner specified above, the components of the actual electrostatic force will be

$$-\frac{dq}{dx}, \quad -\frac{dq}{dy}, \quad -\frac{dq}{dz};$$

and for the average values of these components in the small spaces described above we may write

$$-\frac{d[q]_{\text{Ave}}}{dx}, \quad -\frac{d[q]_{\text{Ave}}}{dy}, \quad -\frac{d[q]_{\text{Ave}}}{dz},$$

for it will make no difference whether we take the average before or after differentiation.

5. The electrodynamic force is determined by the acceleration of electrical flux in all parts of the field, but physicists are not entirely agreed in regard to the laws by which it is determined. This difference of opinion is however of less importance, since it will not affect the result if electrical fluxes are always solenoidal. According to the most simple law, the components of the force are given by the volume-integrals

$$-\iiint\frac{\ddot{\xi}}{r}dv, \quad -\iiint\frac{\ddot{\eta}}{r}dv, \quad -\iiint\frac{\ddot{\zeta}}{r}dv,$$

where dv represents an element of volume, and r the distance of this element from the point for which the value of the electromotive force is to be determined. In other words, the components of the force at any point are determined from the components of acceleration in all parts of the field by the same process by which (in the theories of gravitation, etc.) the value of the potential at any point is determined from the density of matter in all parts of space, except that the sign is to be reversed. Adopting this law, provisionally at least, we may express it by saying that the components of electrodynamic force are equal to the potentials taken negatively of the components of acceleration of electrical flux. And we may write, for brevity,

$$-\text{Pot}\,\ddot{\xi}, \quad -\text{Pot}\,\ddot{\eta}, \quad -\text{Pot}\,\ddot{\zeta},$$

for the components of force, using the symbol *Pot* to denote the operation by which the potential of a mass is derived from its

density. For the average values of these components in the small spaces defined above, we may write

$$-\mathrm{Pot}\,[\ddot{\xi}]_{\mathrm{Ave}}, \qquad -\mathrm{Pot}\,[\ddot{\eta}]_{\mathrm{Ave}}, \qquad -\mathrm{Pot}\,[\ddot{\zeta}]_{\mathrm{Ave}},$$

since it will make no difference whether we take the average before or after the operation of taking the potential.

6. If we write X, Y, Z for the components of the total electromotive force (electrostatic and electrodynamic), we have

$$\left.\begin{aligned}[\mathrm{X}]_{\mathrm{Ave}} &= -\mathrm{Pot}\,[\ddot{\xi}]_{\mathrm{Ave}} - \frac{d[q]_{\mathrm{Ave}}}{dx},\\ &\text{etc.,}\end{aligned}\right\} \tag{3}$$

or by (2)

$$\left.\begin{aligned}[\mathrm{X}]_{\mathrm{Ave}} &= \frac{4\pi^2}{p^2}\,\mathrm{Pot}\,[\xi]_{\mathrm{Ave}} - \frac{d[q]_{\mathrm{Ave}}}{dx},\\ &\text{etc.}\end{aligned}\right\} \tag{4}$$

It will be convenient to represent these relations by a vector notation. If we represent the displacement by $\mathbf{U}$, and the electromotive force by $\mathbf{E}$, the three equations of (3) will be represented by the single vector equation

$$[\mathbf{E}]_{\mathrm{Ave}} = -\mathrm{Pot}\,[\ddot{\mathbf{U}}]_{\mathrm{Ave}} - \nabla[q]_{\mathrm{Ave}}, \tag{5}$$

and the three equations of (4) by the single vector equation

$$[\mathbf{E}]_{\mathrm{Ave}} = \frac{4\pi^2}{p^2}\,\mathrm{Pot}\,[\mathbf{U}]_{\mathrm{Ave}} - \nabla[q]_{\mathrm{Ave}}, \tag{6}$$

where, in accordance with quaternionic usage, $\nabla[q]_{\mathrm{Ave}}$ represents the vector which has for components the derivatives of $[q]_{\mathrm{Ave}}$ with respect to rectangular coordinates. The symbol *Pot* in such a vector equation signifies that the operation which is denoted by this symbol in a scalar equation is to be performed upon each of the components of the vector.

7. We may here observe that if we are not satisfied with the law adopted for the determination of electrodynamic force we have only to substitute for $-Pot$ in these vector equations, and in those which follow, the symbol for the operation, whatever it may be, by which we calculate the electrodynamic force from the acceleration.* For the operation must be of such a character that if the acceleration consist of any number of parts, the force due to the whole acceleration will be the resultant of the forces due to the separate parts. It will evidently make no difference whether we take an average before or after such an operation.

* The same would not be true of the corresponding scalar equations, (3) and (4). For one component of the force might depend upon all the components of acceleration. Such is in fact the case with the law of electromotive force proposed by Weber.

8. Let us now examine the relation which subsists between the values of $[\mathbf{E}]_{\text{Ave}}$ and $[\mathbf{U}]_{\text{Ave}}$ for the same point, that is, between the average electromotive force and the average displacement in a small sphere with its center at the point considered. We have already seen that the forces and the displacements are harmonic functions of the time having a common period.

A little consideration will show that if the average electromotive force in the sphere is given as a function of the time, the displacements in the sphere, both average and actual, must be entirely determined. Especially will this be evident, if we consider that since we have made the radius of the sphere very small in comparison with a wave-length, the average force must have sensibly the same value throughout the sphere (that is, if we vary the position of the center of the sphere for which the average is taken by a distance not greater than the radius, the value of the average will not be sensibly affected), and that the difference of the actual and average force at any point is entirely determined by the motions in the immediate vicinity of that point. If, then, certain oscillatory motions may be kept up in the sphere under the influence of electrostatic and electrodynamic forces due to the motion in the whole field, and if we suppose the motions in and very near that sphere to be unchanged, but the motions in the remoter parts of the field to be altered, only not so as to affect the average resultant of electromotive force in the sphere, the actual resultant of electromotive force will also be unchanged throughout the sphere, and therefore the motions in the sphere will still be such as correspond to the forces.

Now the average displacement is a harmonic function of the time having a period which we suppose given. It is therefore entirely determined for the whole time the vibrations continue by the values of the six quantities

$$[\xi]_{\text{Ave}},\quad [\eta]_{\text{Ave}},\quad [\zeta]_{\text{Ave}},\quad [\dot{\xi}]_{\text{Ave}},\quad [\dot{\eta}]_{\text{Ave}},\quad [\dot{\zeta}]_{\text{Ave}}$$

at any one instant. For the same reason the average electromotive force is entirely determined for the whole time by the values of the six quantities

$$[X]_{\text{Ave}},\quad [Y]_{\text{Ave}},\quad [Z]_{\text{Ave}},\quad [\dot{X}]_{\text{Ave}},\quad [\dot{Y}]_{\text{Ave}},\quad [\dot{Z}]_{\text{Ave}}$$

for the same instant. The first six quantities will therefore be functions of the second, and the principle of the superposition of motions requires that they shall be homogeneous functions of the first degree. And the second six quantities will be homogeneous functions of the first degree of the first six. The coefficients by which these functions are expressed will depend upon the nature of the medium in the vicinity of the point considered. They will also

depend upon the period of vibration, that is, upon the color of the light.*

We may therefore write in vector notation

$$[\mathbf{E}]_{\text{Ave}} = \Phi[\mathbf{U}]_{\text{Ave}} + \Psi[\mathbf{U}]_{\text{Ave}} \tag{7}$$

where Φ and Ψ denote linear functions.†

The optical properties of media are determined by the form of these functions. But all forms of linear functions would not be consistent with the principle of the conservation of energy.

In media which are more or less opaque, and which therefore absorb energy, Ψ must be of such a form that the function always makes an acute angle (or none) with the independent variable. In perfectly transparent media, Ψ must vanish, unless the function is at right angles to the independent variable. So far as is known, the last occurs only when the medium is subject to magnetic influence. In perfectly transparent media, the principle of the conservation of energy requires that Φ should be self-conjugate, i.e., that for three directions at right angles to one another, the function and independent variable should coincide in direction.

In all isotropic media not subject to magnetic influence, it is probable that Φ and Ψ reduce to numerical coefficients, as is certainly the case with Φ for transparent isotropic media.

9. Comparing the two values of $[\mathbf{E}]_{\text{Ave}}$, we have

$$\frac{4\pi^2}{p^2}\,\text{Pot}\,[\mathbf{U}]_{\text{Ave}} - \nabla[q]_{\text{Ave}} = \Phi[\mathbf{U}]_{\text{Ave}} + \Psi[\dot{\mathbf{U}}]_{\text{Ave}}. \tag{8}$$

This equation, in connection with that by which we express the solenoidal character of the displacements, if we regard them as necessarily solenoidal, or in connection with that which expresses the relation between the electrostatic potential and the displacements, if we reject the solenoidal hypothesis, may be regarded as the general equation of the vibrations of monochromatic light, considered as oscillating electrical fluxes. For the symbol *Pot*, however, we must substitute the symbol representing the operation by which electromotive force is calculated from acceleration of flux, with the negative sign, if we are not satisfied with the law provisionally adopted.

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* The relations between the displacements in one of the small spaces considered and the average electromotive force is mathematically analogous to the relation between the displacements in a system of a high degree of complexity and certain forces exerted from without, which are harmonic functions of the time and under the influence of which the system vibrates. The ratio of the displacements to the forces will in general vary with the period, and may vary very rapidly.

An example in which these functions vary very rapidly with the period is afforded by the phenomena of selective absorption and abnormal dispersion.

† A vector is said to be a linear function of another, when the three components of the first are homogeneous functions of the first degree of the three components of the second.

It is important to observe that the existence of molecular vibrations of ponderable matter, due to the passage of light through the medium, will not affect the reasoning by which this equation has been established, provided that the nature and intensity of these vibrations in any small part of the medium (as measured by a wave-length) are entirely determined by the electrical forces and motions in that part of the medium. But the equation would not hold in case of molecular vibrations due to magnetic force. Such vibrations would constitute an oscillating magnetization of the medium, which has already been excluded from the discussion.

The supposition which has sometimes been made,* that electricity possesses a certain mass or inertia, would not at all affect the validity of the equation.

10. The equation may be reduced to a form in some respects more simple by the use of the so-called imaginary quantities. We shall write ι for $\sqrt{(-1)}$. If we differentiate with respect to the time, and substitute $-\frac{4\pi^2}{p^2}[\mathfrak{U}]_{\text{Ave}}$ for $[\ddot{\mathfrak{U}}]_{\text{Ave}}$, we obtain

$$\frac{4\pi^2}{p^2}\operatorname{Pot}[\dot{\mathfrak{U}}]_{\text{Ave}}-\nabla[\dot{q}]_{\text{Ave}}=\Phi[\dot{\mathfrak{U}}]_{\text{Ave}}-\frac{4\pi^2}{p^2}\Psi[\mathfrak{U}]_{\text{Ave}}.$$

If we multiply this equation by ι, either alone or in connection with any real factor, and add it to the preceding, we shall obtain an equation which will be equivalent to the two of which it is formed. Multiplying by $-\frac{p\iota}{2\pi}$ and adding, we have

$$\frac{4\pi^2}{p^2}\operatorname{Pot}\left([\mathfrak{U}]_{\text{Ave}}-\iota\frac{p}{2\pi}[\dot{\mathfrak{U}}]_{\text{Ave}}\right)-\nabla\left([q]_{\text{Ave}}-\iota\frac{p}{2\pi}[\dot{q}]_{\text{Ave}}\right)$$
$$=\left(\Phi+\iota\frac{2\pi}{p}\Psi\right)\left([\mathfrak{U}]_{\text{Ave}}-\iota\frac{p}{2\pi}[\dot{\mathfrak{U}}]_{\text{Ave}}\right)$$

If we set

$$\mathfrak{W}=[\mathfrak{U}]_{\text{Ave}}-\iota\frac{p}{2\pi}[\dot{\mathfrak{U}}]_{\text{Ave}}, \tag{9}$$

$$Q=[q]_{\text{Ave}}-\iota\frac{p}{2\pi}[\dot{q}]_{\text{Ave}}, \tag{10}$$

$$\Theta=\Phi+\iota\frac{2\pi}{p}\Psi, \tag{11}$$

our equation reduces to

$$\frac{4\pi^2}{p^2}\operatorname{Pot}\mathfrak{W}-\nabla Q=\Theta\mathfrak{W}. \tag{12}$$

In this equation Θ denotes a complex linear vector function, i.e., a vector function of which the X-, Y-, and Z-components are expressed in terms of the X-, Y-, and Z-components of the independent variable by means of coefficients of the form $a+\iota b$. $\mathfrak{W}$ is a bivector of which

* See Weber, *Abhandl. d. K. Sächs. Gesellsch. d. Wiss.*, vol. vi, pp. 593-597; Lorberg, *Crelle's Journal*, vol. lxi, p. 55.

the real part represents the averaged displacement $[\mathbf{U}]_{\text{Ave}}$, and the coefficient of ι the rate of increase of the same multiplied by a constant factor. This bivector therefore represents the average state of a small part of the field both with respect to position and velocity. We may also say that the coefficient of ι in $\mathbf{W}$ represents the value of the averaged displacement $[\mathbf{U}]_{\text{Ave}}$ at a time one-quarter of a vibration earlier than the time principally considered.

11. It may serve to fix our ideas to see how $\mathbf{W}$ is expressed as a function of the time. We may evidently set

$$[\mathbf{U}]_{\text{Ave}} = \mathbf{A}_1 \cos\frac{2\pi}{p}t + \mathbf{A}_2 \sin\frac{2\pi}{p}t$$

where $\mathbf{A}_1$ and $\mathbf{A}_2$ are vectors representing the amplitudes of the two parts into which the vibration is resolved. Then

$$\frac{p}{2\pi}[\dot{\mathbf{U}}]_{\text{Ave}} = -\mathbf{A}_1 \sin\frac{2\pi}{p}t + \mathbf{A}_2 \cos\frac{2\pi}{p}t,$$

and

$$[\mathbf{U}]_{\text{Ave}} - \iota\frac{p}{2\pi}[\mathbf{U}]_{\text{Ave}} = (\mathbf{A}_1 - \iota\mathbf{A}_2)\left(\cos\frac{2\pi}{p}t + \iota\sin\frac{2\pi}{p}t\right);$$

that is, if we set $\mathbf{A} = \mathbf{A}_1 - \iota\mathbf{A}_2$,

$$\mathbf{W} = \mathbf{A}e^{\frac{2\pi\iota t}{p}} \tag{13}$$

In like manner we may obtain

$$\mathrm{Q} = ge^{\frac{2\pi\iota t}{p}}, \tag{14}$$

where g is a biscalar, or complex quantity of ordinary algebra. Substituting these values in (12), and cancelling the common factor containing the time, we have

$$\frac{4\pi^2}{p^2}\operatorname{Pot}\mathbf{A} - \nabla g = \Theta\mathbf{A}. \tag{15}$$

Our equation is thus reduced to one between $\mathbf{A}$ and g, and may easily be reduced to one in $\mathbf{A}$ alone.* Now $\mathbf{A}$ represents six numerical quantities (viz., the three components of $\mathbf{A}_1$, and the three of $\mathbf{A}_2$), which may be called the six components of amplitude. The equation, therefore, substantially represents the relations between the six components of amplitude in different parts of the field.† The equation is, however, not really different from (12), since $\mathbf{A}$ and g are only particular values of $\mathbf{W}$ and Q.

* The terms $\nabla\mathrm{Q}$, ∇q are allowed to remain in these equations, because the best manner of eliminating them will depend somewhat upon our admission or rejection of the solenoidal hypothesis.

† The representation of the six components of amplitude by a single letter should not be regarded as an analytical artifice. It only leaves undivided in our notation that which is undivided in the nature of things. The separation of the six components of amplitude is artificial, in that it introduces arbitrary elements into the discussion, viz. the directions of the axes of the coordinates, and the zero of time.

12. From the general equation given above (8, 12, or 15), in connection with the solenoidal hypothesis, we may easily derive the laws of the propagation of plane waves in the interior of a sensibly homogeneous medium, and the laws of reflection and refraction at surfaces between such media. This has been done by Maxwell,* Lorentz,† and others,‡ with fundamental equations more or less similar.

The method, however, by which the fundamental equation has been established in this paper seems free from certain objections which have been brought against the ordinary form of the theory. As ordinarily treated, the phenomena are made to depend entirely on the inductive capacity and the conductivity of the medium, in a manner which may be expressed by the equation

$$[\mathfrak{v}]_{\text{Ave}} = \left(\frac{\mathrm{K}}{4\pi} - \frac{p^2\mathrm{C}}{4\pi^2}\frac{d}{dt}\right)\left(\frac{4\pi^2}{p^2}\operatorname{Pot}[\mathfrak{v}]_{\text{Ave}} - \nabla[q]_{\text{Ave}}\right), \tag{16}$$

which will be equivalent to (12), if

$$\mathfrak{w} = \left(\frac{\mathrm{K}}{4\pi} - \iota\frac{p\mathrm{C}}{2\pi}\right)\left(\frac{4\pi^2}{p^2}\operatorname{Pot}\mathfrak{w} - \nabla\mathrm{Q}\right), \tag{17}$$

where K and C denote in the most general case the linear vector functions, but in isotropic bodies the numerical coefficients, which represent inductive capacity and conductivity. By a simple transformation {see (9) and (10)}, this equation becomes

$$\Theta^{-1} = \frac{\mathrm{K}}{4\pi} - \iota\frac{p\mathrm{C}}{2\pi}, \tag{18}$$

where Θ^{-1} represents the function inverse to Θ.

Now, while experiment appears to verify the existence of such a law as is expressed by equation (12), it does not show that Θ has the precise form indicated by equation (16). In other words, experiment does not satisfactorily verify the relations expressed by (16) and (17), if K and C are understood to be the operators (or, in isotropic bodies, the numbers) which represent inductive capacity and conductivity in the ordinary sense of the terms.

The discrepancy is most easily shown in the most simple case, when the medium is isotropic and perfectly transparent, and Θ reduces to a numerical quantity. The square of the velocity of plane waves is then equal to $\frac{\Theta}{4\pi}$, and equation (18) would make it independent of the

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* *Phil. Trans.*, vol. clv (1865), p. 459, or *Treatise on Electricity and Magnetism*, chap. xx.

† Schlömilch's *Zeitschrift*, vol. xxii, pp. 1-30 and 205-219; xxiii, pp. 197-210.

‡ See Fitzgerald, *Phil. Trans.*, vol. clxxi, p. 691; J. J. Thomson, *Phil. Mag.*, (5), vol. ix, p. 284; Rayleigh, *Phil. Mag.* (5), vol. xii, p. 81.

That the electromagnetic theory of light gives the conditions relative to the boundary of different media, which are required by the phenomena of reflection and refraction, was first shown by Helmholtz. See *Crelle's Journal*, vol. lxxii (1870), p. 57.

period; that is, would give no dispersion of colors. The case is essentially the same in transparent bodies which are not isotropic.*

The case is worse with metals, which are characterized electrically by great conductivity, and optically by great opacity. In their papers cited above, Lorentz and Rayleigh have observed that the experiments of Jamin on the reflection of light from metallic surfaces would often require, as ordinarily interpreted on the electromagnetic theory, a negative value for the inductive capacity of the metal. This would imply that the electrical equilibrium in the metal is unstable. The objection, therefore, is essentially the same as that which Lord Rayleigh had previously made to Cauchy's theory of metallic reflection, viz., that the apparent mechanical explanation of the phenomena is illusory, since the numerical values given by experiment as interpreted on Cauchy's theory would involve an unstable equilibrium of the ether in the metal.†

13. All this points to the same conclusion—that the ordinary view of the phenomena is inadequate. The object of this paper will be accomplished, if it has been made clear how a point of view more in accordance with what we know of the molecular constitution of bodies will give that part of the ordinary theory which is verified by experiment, without including that part which is in opposition to observed facts.‡

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* See note to the first paper of Lorentz, cited above, Schlömilch, vol. xxii, p. 23.

† See *Phil. Mag.* (4), vol. xliii (1872), p. 321.

‡ The consideration of the processes which we may suppose to take place in the smallest parts of a body through which light is transmitted, farther than is necessary to establish the general equation given above, is foreign to the design of this paper. Yet a word may be added with respect to the difficulties signalized in the ordinary form of the theory. The comparatively simple case of a perfectly transparent body has been examined more in detail in one of the papers already cited, where there is given an explanation of the dispersion of colors from the point of view of this paper. It is there shown that the effect of the non-homogeneity of the body in its smallest parts is to add a term to the expression for the kinetic energy of electrical waves, which for an isotropic body may be roughly described as similar to that which would be required if the electricity had a certain mass or inertia. (See especially §§ 7, 9 and 12, [this volume pages 185 ff.]) The same must be true of media of any degree of opacity. Now the difficulty with the optical properties of the metals is that the real part of Θ (or Θ^{-1}) is in some cases negative. This implies that at a moment of greatest displacement the electromotive force is in the direction opposite to the displacement, instead of having the same direction, as in transparent isotropic bodies. Now a certain part of the electromotive force must be required to oppose the apparent inertia, and another part to oppose the electrical elasticity of the medium. These parts of the force must have opposite directions. In transparent bodies the latter part is by far the greater. But it need not surprise us that the former should be the greater in some metals.

It has been remarked by Lorentz that the difficulty with respect to metals would be in a measure relieved if we should suppose electricity to have the property of inertia. (See § 11 of his third paper, Schlömilch's *Zeitschrift*, vol. xxiii, p. 208.) But a supposition of this kind, taken literally, would involve a dispersion of colors in vacuo, and still be inadequate, as Lorentz remarks, to explain the phenomena observed in metals.

While the writer has aimed at a greater degree of rigor than is usual in the establishment of the fundamental equation of monochromatic light, it is not claimed that this equation is absolutely exact. The contrary is evident from the fact that the equation does not embrace the phenomena which characterize such circularly polarizing bodies as quartz. This, however, only implies the neglect of extremely small quantities—very small, for example, as compared with those which determine the dispersion of colors. In one of the papers already cited,* the case of a perfectly transparent body is treated with a higher degree of approximation, so as to embrace the phenomena in question.

* See page 195 of this volume.

XIV.

A COMPARISON OF THE ELASTIC AND THE ELECTRICAL THEORIES OF LIGHT WITH RESPECT TO THE LAW OF DOUBLE REFRACTION AND THE DISPERSION OF COLORS.

[*American Journal of Science*, ser. 3, vol. XXXV, pp. 467–475, June, 1888.]

It is claimed for the electrical * theory of light that it is free from serious difficulties, which beset the explanation of the phenomena of light by the dynamics of elastic solids. Just what these difficulties are, and why they do not occur in the explanation of the same phenomena by the dynamics of electricity, has not perhaps been shown with all the simplicity and generality which might be desired. Such a treatment of the subject is however the more necessary on account of the ever-increasing bulk of the literature on either side, and the confusing multiplicity of the elastic theories. It is the object of this paper to supply this want, so far as respects the propagation of plane waves in transparent and sensibly homogeneous media. The simplicity of this part of the subject renders it appropriate for the first test of any optical theory, while the precision of which the experimental determinations are capable, renders the test extremely rigorous.

It is moreover, as the writer believes, an appropriate time for the discussion proposed, since on one hand the experimental verification of Fresnel's Law has recently been carried to a degree of precision far exceeding anything which we have had before,† and on the other, the

* The term *electrical* seems the most simple and appropriate to describe that theory of light which makes it consist in electrical motions. The cases in which any distinctively magnetic action is involved in the phenomena of light are so exceptional, that it is difficult to see any sufficient reason why the general theory should be called *electromagnetic*, unless we are to call all phenomena electromagnetic which depend on the motions of electricity.

† In the recent experiments of Professor Hastings relating to the index of refraction of the extraordinary ray in Iceland spar for the spectral line D_2 and a wave-normal inclined at about 31° to the optic axis, the difference between the observed and the calculated values was only two or three units in the sixth decimal place (in the seventh significant figure), which was about the probable error of the determinations. See *Am. Jour. Sci.* ser. 3, vol. XXXV, p. 60.

discovery of a remarkable theorem relating to the vibrations of a strained solid* has given a new impulse to the study of the elastic theory of light.

Let us first consider the facts to which a correct theory must conform.

It is generally admitted that the phenomena of light consist in motions (of the type which we call wave-motions) of something which exists both in space void of ponderable matter, and in the spaces between the molecules of bodies, perhaps also in the molecules themselves. The kinematics of these motions is pretty well understood; the question at issue is whether it agrees with the dynamics of elastic solids or with the dynamics of electricity.

In the case of a simple harmonic wave-motion, which alone we need consider, the wave-velocity (V) is the quotient of the wave-length (l) by the period of vibration (p). These quantities can be determined with extreme accuracy. In media which are sensibly homogeneous but not isotropic the wave-velocity V, for any constant value of the period, is a quadratic function of the direction cosines of a certain line, viz., the normal to the so-called "plane of polarization." The physical characteristics of this line have been a matter of dispute. Fresnel considered it to be the direction of displacement. Others have maintained that it is the common perpendicular to the wave-normal and the displacement. Others again would define it as that component of the displacement which is perpendicular to the wave-normal. This of course would differ from Fresnel's view only in case the displacements are not perpendicular to the wave-normal, and would in that case be a necessary modification of his view. Although this dispute has been one of the most celebrated in physics, it seems to be at length substantially settled, most directly by experiments upon the scattering of light by small particles, which seems to show decisively that in isotropic media at least the displacements are normal to the "plane of polarization," and also, with hardly less cogency, by the difficulty of accounting for the intensities of reflected and refracted light on any other

* Sir Wm. Thomson has shown that if an elastic incompressible solid in which the potential energy of any homogeneous strain is proportional to the sum of the squares of the reciprocals of the principal elongations *minus* three is subjected to any homogeneous strain by forces applied to its surface, the transmission of plane waves of distortion, superposed on this homogeneous strain, will follow exactly Fresnel's law (including the direction of displacement), the three principal velocities being proportional to the reciprocals of the principal elongations. It must be a surprise to mathematicians and physicists to learn that a theorem of such simplicity and beauty has been waiting to be discovered in a field which has been so carefully gleaned. See page 116 of the current volume (xxv) of the *Philosophical Magazine*.

supposition.* It should be added that all diversity of opinion on this subject has been confined to those whose theories are based on the dynamics of elastic bodies. Defenders of the electrical theory have always placed the electrical displacement at right angles to the "plane of polarization." It will, however, be better to assume this direction of the displacement as probable rather than as absolutely certain, not so much because many are likely to entertain serious doubts on the subject, as in order not to exclude views which have at least a historical interest.

The wave-velocity, then, for any constant period, is a quadratic function of the cosines of a certain direction, which is probably that of the displacement, but in any case determined by the displacement and the wave-normal. The coefficients of this quadratic function are functions of the period of vibration. It is important to notice that these coefficients vary separately, and often quite differently, with the period, and that the case does not at all resemble that of a quadratic function of the direction-cosines multiplied by a quantity depending on the period.

In discussing the dynamics of the subject we may gain something in simplicity by considering a system of stationary waves, such as results from two similar systems of progressive waves moving in opposite directions. In such a system the energy is alternately entirely kinetic and entirely potential. Since the total energy is constant, we may set the average kinetic energy per unit of volume at the moment when there is no potential energy, equal to the average potential energy per unit of volume when there is no kinetic energy.† We may call this the equation of energies. It will contain the quantities l and p, and thus furnish an expression for the velocity of either system of progressive waves. We have to see whether the elastic or the electric theory gives the expression most conformed to the facts.

Let us first apply the elastic theory to the case of the so-called

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* "At the same time, if the above reasoning be valid, the question as to the direction of the vibrations in polarized light is decided in accordance with the view of Fresnel. . . . I confess I cannot see any room for doubt as to the result it leads to. . . . I only mean that *if* light, as is generally supposed, consists of transversal vibrations similar to those which take place in an elastic solid, the vibration must be normal to the plane of polarization." Lord Rayleigh "On the Light from the Sky, its Polarization and Color;" *Phil. Mag.* (4), xli (1871), p. 109.

"Green's dynamics of polarization by reflexion, and Stokes' dynamics of the diffraction of polarized light, and Stokes' and Rayleigh's dynamics of the blue sky, all agree in, as it seems to me, irrefragably, demonstrating Fresnel's original conclusion, that in plane polarized light the line of vibration is perpendicular to the plane of polarization." Sir Wm. Thomson, *loc. citat.*

† The terms *kinetic energy* and *potential energy* will be used in this paper to denote these average values.

vacuum. If we write h for the amplitude measured in the middle between two nodal planes, the velocities of displacement will be as $\frac{h}{p}$, and the kinetic energy will be represented by $A\frac{h^2}{p^2}$, where A is a constant depending on the density of the medium. The potential energy, which consists in distortion of the medium, may be represented by $B\frac{h^2}{l^2}$, where B is a constant depending on the rigidity of the medium. The equation of energies, on the elastic theory, is therefore

$$A\frac{h^2}{p^2}=B\frac{h^2}{l^2} \tag{1}$$

which gives

$$V^2=\frac{l^2}{p^2}=\frac{B}{A}. \tag{2}$$

In the electrical theory, the kinetic energy is not determined by the simple formula of ordinary dynamics from the square of the velocity of each element, but is found by integrating the product of the velocities of each pair of elements divided by the distance between them. Very elementary considerations suffice to show that a quantity thus determined when estimated per unit of volume will vary as the square of the wave-length. We may therefore set $Fl^2\frac{h^2}{p^2}$ for the kinetic energy, F being a constant. The potential energy does not consist in distortion of the medium, but depends upon an elastic resistance to the separation of the electricities, which constitutes the electrical displacement, and is proportioned to the square of this displacement. The average value of the potential energy per unit of volume will therefore be represented in the electrical theory by Gh^2, where G is a constant, and the equation of energies will be

$$Fl^2\frac{h^2}{p^2}=Gh^2 \tag{3}$$

which gives

$$V^2=\frac{l^2}{p^2}=\frac{G}{F}. \tag{4}$$

Both theories give a constant velocity, as is required. But it is instructive to notice the profound difference in the equations of energy from which this result is derived. In the elastic theory the square of the wave-length appears in the potential energy as a divisor; in the electrical theory it appears in the kinetic energy as a factor.

Let us now consider how these equations will be modified by the presence of ponderable matter, in the most general case of transparent and sensibly homogeneous bodies. This subject is rendered much more simple by the fact that the distances between the ponderable molecules are very small compared with a wave-length. Or, what

amounts to the same thing, but may present a more distinct picture to the imagination, the wave-length may be regarded as enormously great in comparison with the distances between neighboring molecules. Whatever view we take of the motions which constitute light, we can hardly suppose them (disturbed as they are by the presence of the ponderable molecules) to be in strictness represented by the equations of wave-motion. Yet in a certain sense a wave-motion may and does exist. If, namely, instead of the actual displacement at any point, we consider the average displacement in a space large enough to contain an immense number of molecules, and yet small as measured by a wave-length, such average displacements may be represented by the equations of wave-motion; and it is only in this sense that any theory of wave-motion can apply to the phenomena of light in transparent bodies. When we speak of displacements, amplitudes, velocities (of displacement), etc., it must therefore be understood in this way.

The actual kinetic energy, on either theory, will evidently be greater than that due to the motion thus averaged or smoothed, and to a degree presumably depending on the direction of the displacement. But since displacement in any direction may be regarded as compounded of displacements in three fixed directions, the additional energy will be a quadratic function of the components of velocity of displacement, or, in other words, a quadratic function of the direction-cosines of the displacement multiplied by the square of the amplitude and divided by the square of the period.* This additional energy may be understood as including any part of the kinetic energy of the wave-motion which may belong to the ponderable particles. The term to be added to the kinetic energy on the electric theory may therefore be written $f_D \frac{h^2}{p^2}$, where f_D is a quadratic function of the direction-cosines of the displacement. The elastic theory requires a term of precisely the same character, but since the term to which it is to be added is of the same general form, the two may be incorporated in a single term of the form $A_D \frac{h^2}{p^2}$, where A_D is a quadratic function of the direction-cosines of the displacement. We must, however, notice that both A_D and f_D are not entirely independent of the period. For the manner in which the flux of the luminiferous medium is distributed among the ponderable molecules will naturally depend somewhat upon the period. The same is true of the degree to which the molecules may be thrown into vibration. But A_D and f_D will be independent of the wave-length (except so far as this is

* For proof *in extenso* of this proposition, when the motions are supposed electrical, the reader is referred to page 187 of this volume.

connected with the period), because the wave-length is enormously great compared with the size of the molecules and the distances between them.

The potential energy on the elastic theory must be increased by a term of the form $b_D h^2$, where b_D is a quadratic function of the direction-cosines of the displacement. For the ponderable particles must oppose a certain elastic resistance to the displacement of the ether, which in æolotropic bodies will presumably be different in different directions. The potential energy on the electric theory will be represented by a single term of the same form, say $G_D h^2$, where a quadratic function of the direction-cosines of the displacement, G_D, takes the place of the constant G, which was sufficient when the ponderable particles were absent. Both G_D and b_D will vary to some extent with the period, like A_D and f_D, and for the same reason.

In regard to that potential energy, which on the elastic theory is independent of the direct action of the ponderable molecules, it has been supposed that in æolotropic bodies the effect of the molecules is such as to produce an æolotropic state in the ether, so that the energy of a distortion varies with its orientation. This part of the potential energy will then be represented by $B_{ND} \frac{h^2}{l^2}$, where B_{ND} is a function of the directions of the wave-normal and the displacement. It may easily be shown that it is a quadratic function both of the direction-cosines of the wave-normal and of those of the displacement. Also, that if the ether in the body when undisturbed is not in a state of stress due to forces at the surface of the body, or if its stress is uniform in all directions, like a hydrostatic pressure, the function B_{ND} must be symmetrical with respect to the two sets of direction-cosines.

The equation of energies for the elastic theory is therefore

$$A_D \frac{h^2}{p^2} = B_{ND} \frac{h^2}{l^2} + b_D h^2, \tag{5}$$

which gives

$$V^2 = \frac{l^2}{p^2} = \frac{B_{ND}}{A_D - b_D p^2}. \tag{6}$$

The equation of energies for the electrical theory is

$$F l^2 \frac{h^2}{p^2} + f_D \frac{h^2}{p^2} = G_D h^2, \tag{7}$$

which gives

$$V^2 = \frac{l^2}{p^2} = \frac{G_D}{F} - \frac{f_D}{F p^2}. \tag{8}$$

It is evident at once that the electrical theory gives exactly the form that we want. For any constant period the square of the wave-velocity is a quadratic function of the direction-cosines of the displacement. When the period varies, this function varies,

the different coefficients in the function varying separately, because G_D and f_D will not in general be similar functions.* If we consider a constant direction of displacement while the period varies, G_D and f_D will only vary so far as the type of the motion varies, i.e., so far as the manner in which the flux distributes itself among the ponderable molecules and intermolecular spaces, and the extent to which the molecules take part in the motion are changed. There are cases in which these vary rapidly with the period, viz., cases of selective absorption and abnormal dispersion. But we may fairly expect that there will be many cases in which the character of the motion in these respects will not vary much with the period. $\frac{G_D}{F}$ and $\frac{f_D}{F}$ will then be sensibly constant and we have an approximate expression for the general law of dispersion, which agrees remarkably well with experiment.†

If we now return to the equation of energies obtained from the elastic theory, we see at once that it does not suggest any such relation as experiment has indicated, either between the wave-velocity and the direction of displacement, or between the wave-velocity and the period. It remains to be seen whether it can be brought to agree with experiment by any hypothesis not too violent.

In order that V^2 may be a quadratic function of any set of direction-cosines, it is necessary that A_D and b_D shall be independent of the direction of the displacement, in other words, in the case of a crystal like Iceland spar, that the direct action of the ponderable molecules upon the ether, shall affect both the kinetic and the potential energy in the same way, whether the displacement take place in the direction of the optic axis or at right angles to it. This is contrary to everything which we should expect. If, nevertheless, we make this supposition, it remains to consider B_{ND}. This must be a quadratic function of a certain direction, which is almost certainly that of the displacement. If the medium is free from external stress (other than hydrostatic), B_{ND}, as we have seen, is symmetrical with respect to the wave-normal and the direction of displacement, and a quadratic function of the direction-cosines of each. The only single direction of which it can be a function is the common perpendicular to these two directions. If the wave-normal and the displacement are perpendicular, the direction-cosines

* But G_D, f_D, and V^2, considered as functions of the direction of displacement, are all subject to any law of symmetry which may belong to the structure of the body considered. The resulting optical characteristics of the different crystallographic systems are given on pages 192-194.

† This will appear most distinctly if we consider that V divided by the velocity of light *in vacuo* gives the reciprocal of the index of refraction, and p multiplied by the same quantity gives the wave-length *in vacuo*.

of the common perpendicular to both will be linear functions of the direction-cosines of each, and a quadratic function of the direction-cosines of the common perpendicular will be a quadratic function of the direction-cosines of each. We may thus reconcile the theory with the law of double refraction, in a certain sense, by supposing that A_D and b_D are independent of the direction of displacement, and that B_{ND} and therefore V^2 is a quadratic function of the direction-cosines of the common perpendicular to the wave-normal and the displacement. But this supposition, besides its intrinsic improbability so far as A_D and b_D are concerned, involves a direction of the displacement which is certainly or almost certainly wrong.

We are thus driven to suppose that the undisturbed medium is in a state of stress, which, moreover, is not a simple hydraulic stress. In this case, by attributing certain definite physical properties to the medium, we may make the function B_{ND} become independent of the direction of the wave-normal, and reduce to a quadratic function of the direction-cosines of the displacement.* This entirely satisfies Fresnel's Law, including the direction of displacement, if we can suppose A_D and b_D independent of the direction of displacement. But this supposition, in any case difficult for aeolotropic bodies, seems quite irreconcilable with that of a permanent (not hydrostatic) stress.

For this stress can only be kept up by the action of the ponderable molecules, and by a sort of action which hinders the passage of the ether past the molecules. Now the phenomena of reflection and refraction would be very different from what they are, if the optical homogeneity of a crystal did not extend up very close to the surface. This implies that the stress is produced by the ponderable particles in a very thin lamina at the surface of the crystal, much less in thickness, it would seem probable, than a wave-length of yellow light. And this again implies that the power of the ponderable particles to pin down the ether, as it were, to a particular position is very great, and that the term in the energy relating to the motion of the ether relative to the ponderable particles is very important. This is the term containing the factor b_D, which it is difficult to suppose independent of the direction of displacement because the dimensions and arrangement of the particles are different in different directions. But our present hypothesis has brought in a new reason for supposing b_D to depend on the direction of displacement, viz., on account of the stress of the medium. A general displacement of the medium midway between two nodal planes, when it is restrained at innumerable points by the ponderable particles, will

* See note on page 224.

produce special distortions due to these particles. The nature of these distortions is wholly determined by the direction of displacement, and it is hard to conceive of any reason why the energy of these distortions should not vary with the direction of displacement, like the energy of the general distortion of the wave-motion, which is partly determined by the displacement and partly by the wave-normal.*

But the difficulties of the elastic theory do not end with the law of double refraction, although they are there more conspicuous on account of the definite and simple law by which they can be judged. It does not easily appear how the equation of energies can be made to give anything like the proper law of the dispersion of colors. Since for given directions of the wave-normal and displacement, or in an isotropic body, B_{ND} is constant, and also A_D and b_D, except so far as the type of the vibration varies, the formula requires that the square of the index of refraction (which is inversely as V^2) should be equal to a constant diminished by a term proportional to the square of the period, except so far as this law is modified by a variation of the type of vibration. But experiment shows nothing like this law. Now the variation in the type of vibration is sometimes very important,—it plays the leading rôle in the phenomena of selective absorption and abnormal dispersion,—but this is certainly not always the case. It seems hardly possible to suppose that the type of vibration is always so variable as entirely to mask the law which is indicated by the formula when A_D and b_D (with B_{ND}) are regarded as constant. This is especially evident when we consider that the effect on the wave-velocity of a small variation in the type of vibration will be a small quantity of the second order.†

The phenomena of dispersion, therefore, corroborate the conclusion which seemed to follow inevitably from the law of double refraction alone.

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* The reader may perhaps ask how the above reasoning is to be reconciled with the fact that the law of double refraction has been so often deduced from the elastic theory. The troublesome terms are b_D and the variable part of A_D, which express the direct action of the ponderable molecules on the ether. So far as the (quite limited) reading and recollection of the present writer extend, those who have sought to derive the law of double refraction from the theory of elastic solids have generally either neglected this direct action—a neglect to which Professor Stokes calls attention more than once in his celebrated "Report on Double Refraction" (*Brit. Assoc.*, 1862, pp. 264, 268)—or taking account of this action they have made shipwreck upon a law different from Fresnel's and contradicted by experiment.

† See pages 190, 191 of this volume, or Lord Rayleigh's *Theory of Sound*, vol. i, p. 84.

XV.

A COMPARISON OF THE ELECTRIC THEORY OF LIGHT AND SIR WILLIAM THOMSON'S THEORY OF A QUASI-LABILE ETHER.

[*American Journal of Science*, ser. 3, vol. XXXVII, pp. 139-144, February, 1889.]

A REMARKABLE paper by Sir William Thomson, in the November number of the *Philosophical Magazine*, has opened a new vista in the possibilities of the theory of an elastic ether. Since the general theory of elasticity gives three waves characterized by different directions of displacement for a single wave-plane, while the phenomena of optics show but two, the first point in accommodating any theory to observation, is to get rid (absolutely or sensibly) of the third wave. For this end, it has been common to make the ether incompressible, or, as it is sometimes expressed, to make the velocity of the third wave infinite. The velocity of the wave of compression becomes in fact infinite as the compressibility vanishes. Of course it has not escaped the notice of physicists that we may also get rid of the third wave by making its velocity zero, as may be done by giving certain values to the constants which express the elastic properties of the medium, but such values have appeared impossible, as involving an unstable state of the medium. The condition of incompressibility, absolute or approximate, has therefore appeared necessary.* This question of instability has now, however, been subjected to a more searching examination, with the result that the instability does not really exist "*provided we either suppose the medium to extend all through boundless space, or give it a fixed containing vessel as its boundary.*" This renders possible a very simple theory of light, which has been shown to give Fresnel's laws for the intensities of reflected and refracted light and for double refraction, so far as concerns the phenomena which can be directly observed. The displacement in an aeolotropic medium is in the same plane passing through the wave-normal as was supposed by Fresnel,

* It was under this impression that the paper entitled "A Comparison of the Elastic and the Electric Theories of Light with respect to the Law of Double Refraction and the Dispersion of Colors," [this volume pp. 223-231], was written. The conclusions of that paper, except so far as respects the dispersion of colors, will not apply to the new theory.

but its position in that plane is different, being perpendicular to the ray instead of to the wave-normal.*

It is the object of this paper to compare this new theory with the electric theory of light. In the limiting cases, that is, when we regard the velocity of the missing wave in the elastic theory as zero, and in the electric theory as infinite, we shall find a remarkable correspondence between the two theories, the motions of monochromatic light within isotropic or aeolotropic media of any degree of transparency or opacity, and at the boundary between two such media, being represented by equations absolutely identical, except that the symbols which denote displacement in one theory denote force in the other, and *vice versâ*.† In order to exhibit this correspondence completely and clearly, it is necessary that the fundamental principles of the two theories should be treated with the same generality, and, so far as possible, by the same method. The immediate consequences of the new theory will therefore be deduced with the same generality and essentially by the same method which has been used with reference to the electric theory in a former volume of this Journal [page 211 of this volume].

The elastic properties of the ether, according to the new theory, *in its limiting case*, may be very simply expressed by means of a vector operator, for which we shall use Maxwell's designation. The *curl* of a vector is defined to be another vector so derived from the first that if u, v, w be the rectangular components of the first, and u', v', w', those of its curl,

$$u' = \frac{dw}{dy} - \frac{dv}{dz}, \quad v' = \frac{du}{dz} - \frac{dw}{dx}, \quad w' = \frac{dv}{dx} - \frac{du}{dy}, \tag{1}$$

where x, y, z are rectangular coordinates. With this understanding, if the displacement of the ether is represented by the vector $\mathfrak{E}$, the force exerted upon any element by the surrounding ether will be

$$-\mathrm{B}\ \mathrm{curl}\ \mathrm{curl}\ \mathfrak{E}\ dx\ dy\ dz, \tag{2}$$

where B is a scalar (the so-called *rigidity* of the ether) having the same constant value throughout all space, whether ponderable matter is present or not.

Where there is no ponderable matter, this force must be equated to the reaction of the inertia of the ether. This gives, with omission of the common factor $dx\ dy\ dz$,

$$\mathrm{A}\ddot{\mathfrak{E}} = -\mathrm{B}\ \mathrm{curl}\ \mathrm{curl}\ \mathfrak{E}, \tag{3}$$

where A denotes the density of the ether.

* Sir William Thomson, *loc. citat.* R. T. Glazebrook, *Phil. Mag.*, December, 1888.

† In giving us a new interpretation of the equations of the electric theory, the author of the new theory has in fact enriched the mathematical theory of physics with something which may be compared to the celebrated *principle of duality* in geometry.

The presence of ponderable matter disturbs the motions of the ether, and renders them too complicated for us to follow in detail. Nor is this necessary, for the quantities which occur in the equations of optics represent average values, taken over spaces large enough to smooth out the irregularities due to the ponderable particles, although very small as measured by a wave-length.* Now the general principles of harmonic motion† show that to maintain in any element of volume the motion represented by

$$\mathfrak{E} = \mathfrak{A} e^{2\pi i \frac{t}{p}}, \tag{4}$$

$\mathfrak{A}$ being a complex vector constant, will require a force from outside represented by a complex linear vector function of $\mathfrak{E}$, that is, the three components of the force will be complex linear functions of the three components of $\mathfrak{E}$. We shall represent this force by

$$B\Psi\ddot{\mathfrak{E}}\, dx\, dy\, dz, \tag{5}$$

where Ψ represents a complex linear vector function.‡

If we now equate the force required to maintain the motion in any element to that exerted upon the element by the surrounding ether, we have the equation

$$\Psi\ddot{\mathfrak{E}} = -\operatorname{curl}\operatorname{curl}\mathfrak{E}, \tag{6}$$

which expresses the general law for the motion of monochromatic light within any sensibly homogeneous medium, and may be regarded as implicitly including the conditions relating to the boundary of two such media, which are necessary for determining the intensities of reflected and refracted light.

For let $u,\ v,\ w$ be the components of $\mathfrak{E}$,

$u',\ v',\ w'$ „ „ curl $\mathfrak{E}$,

$u'',\ v'',\ w''$ „ „ curl curl $\mathfrak{E}$,

so that

$$u' = \frac{dw}{dy} - \frac{dv}{dz}, \quad v' = \frac{du}{dz} - \frac{dw}{dx}, \quad w' = \frac{dv}{dx} - \frac{du}{dy},$$

$$u'' = \frac{dw'}{dy} - \frac{dv'}{dz}, \quad v'' = \frac{du'}{dz} - \frac{dw'}{dx}, \quad w'' = \frac{dv'}{dx} - \frac{du'}{dy};$$

and let the interface be perpendicular to the axis of Z. It is evident

* This is in no respect different from what is always tacitly understood in the theory of sound, where the displacements, velocities, densities considered are always such average values. But in the theory of light, it is desirable to have the fact clearly in mind on account of the two interpenetrating media (imponderable and ponderable), the laws of light not being in all respects the same as they would be for a single homogeneous medium.

† See Lord Rayleigh's *Theory of Sound*, vol. i, chapters iv, v.

‡ It amounts essentially to the same thing, whether we regard the force as a linear vector function of $\mathfrak{E}$ or of $\ddot{\mathfrak{E}}$, since these differ only by the constant factor $-\frac{4\pi^2}{p^2}$. But there are some advantages in expressing the force as a function of $\ddot{\mathfrak{E}}$, because the greater part of the force, in the most important cases, is required to overcome the inertia of the ether, and is thus more immediately connected with $\ddot{\mathfrak{E}}$.

that if u' or v' is discontinuous at the interface, the value of u'' or v'' becomes in a sense infinite, i.e., curl curl $\mathfrak{E}$, and therefore by (6) $\Psi\ddot{\mathfrak{E}}$, will be infinite. Now both $\mathfrak{E}$ and Ψ are discontinuous at the interface, but infinite values for $\Psi\ddot{\mathfrak{E}}$ are not admissible. Therefore u' and v' are continuous. Again, if u or v is discontinuous, u' or v' will become infinite, and therefore u'' or v''. Therefore u and v are continuous. These conditions may be expressed in the most general manner by saying that the components of $\mathfrak{E}$ and curl $\mathfrak{E}$ parallel to the interface are continuous. This gives four complex scalar conditions, or in all eight scalar conditions, for the motion at the interface, which are sufficient to determine the amplitude and phase of the two reflected and the two refracted rays in the most general case. It is easy, however, to deduce from these four complex conditions, two others, which are interesting and sometimes convenient. It is evident from the definitions of w' and w'' that if u, v, u', and v' are continuous at the interface w' and w'' will also be continuous. Now $-w''$ is equal to the component of $\Psi\ddot{\mathfrak{E}}$ normal to the interface. The following quantities are therefore continuous at the interface:

$$\left.\begin{array}{ll} \text{the components parallel to the interface of} & \mathfrak{E}, \\ \text{the component normal to the interface of} & \Psi\mathfrak{E}, \\ \text{all components of} & \text{curl } \mathfrak{E}. \end{array}\right\} \qquad (7)$$

To compare these results with those derived from the electrical theory, we may take the general equation of monochromatic light on the electrical hypothesis from a paper in a former volume of this Journal. This equation, which with an unessential difference of notation may be written *

$$-\text{Pot}\,\ddot{\mathfrak{F}} - \nabla Q = 4\pi\Phi\mathfrak{F}, \qquad (8)$$

was established by a method and considerations similar to those which have been used to establish equation (6), except that the ordinary law of electrodynamic induction had the place of the new law of elasticity. $\mathfrak{F}$ is a complex vector representing the electrical displacement as a harmonic function of the time; Φ is a complex linear vector operator, such that $4\pi\Phi\mathfrak{F}$ represents the electromotive force necessary to keep up the vibration $\mathfrak{F}$. Q is a complex scalar representing the electrostatic potential, ∇Q the vector of which the three components are

$$\frac{dQ}{dx}, \quad \frac{dQ}{dy}, \quad \frac{dQ}{dz}.$$

Pot denotes the operation by which in the theory of gravitation the potential is calculated from the density of matter.† When it is

* See page 218 of this volume, equation (12).

† The symbol – Pot is therefore equivalent to $4\pi\nabla^{-2}$, as used by Sir William Thomson (with a happy economy of symbols) at the last meeting of British Association to express the same law of electrodynamic induction, except that the symbol is here used as a vector operator. See *Nature*, vol. xxxviii, p. 571, *sub init.*

applied as here to a vector, the three components of the result are to be calculated separately from the three components of the operand. $-\nabla Q$ is therefore the electrostatic force, and $-\text{Pot}\,\ddot{\mathfrak{F}}$ the electrodynamic force. In establishing the equation, it was not assumed that the electrical motions are *solenoidal*, or such as to satisfy the so-called "equation of continuity." We may now, however, make this assumption, since it is the extreme case of the electric theory which we are to compare with the extreme case of the elastic.

It results from the definitions of *curl* and ∇ that curl $\nabla Q = 0$. We may therefore eliminate Q from equation (8) by taking the curl. This gives

$$-\text{curl Pot}\,\ddot{\mathfrak{F}} = 4\pi\,\text{curl}\,\Phi\mathfrak{F}. \tag{9}$$

Since curl curl and $\frac{1}{4\pi}$ Pot are inverse operators for solenoidal vectors, we may get rid of the symbol Pot by taking the curl again. We thus get

$$-\ddot{\mathfrak{F}} = \text{curl curl}\,\Phi\mathfrak{F}. \tag{10}$$

The conditions for the motion at the boundary between different media are easily obtained from the following considerations. Pot $\ddot{\mathfrak{F}}$ and Q are evidently continuous at the interface. Therefore the components parallel to the interface of ∇Q, and by (8) of $\Phi\mathfrak{F}$, will be continuous. Again, curl Pot $\ddot{\mathfrak{F}}$ is continuous at the interface, as appears from the consideration that curl Pot $\dot{\mathfrak{F}}$ is the magnetic force due to the electrical motions $\dot{\mathfrak{F}}$. Therefore, by (9), curl $\Phi\mathfrak{F}$ is continuous. The solenoidal condition requires that the component of $\mathfrak{F}$ normal to the interface shall be continuous.

The following quantities are therefore continuous at the interface:

$$\left.\begin{array}{ll} \text{the components parallel to the interface of} & \Phi\mathfrak{F}, \\ \text{the component normal to the interface of} & \mathfrak{F}, \\ \text{all components of} & \text{curl}\,\Phi\mathfrak{F}. \end{array}\right\} \tag{11}$$

Of these conditions, the two relating to the normal components of $\mathfrak{F}$ and curl $\Phi\mathfrak{F}$ are easily shown to result from the other four conditions, as in the analogous case in the elastic theory.

If we now compare in the two theories the differential equations of the motion of monochromatic light for the interior of a sensibly homogeneous medium, (6) and (10), and the special conditions for the boundary between two such media as represented by the continuity of the quantities (7) and (11), we find that these equations and conditions become identical, if

$$\mathfrak{F} = \Psi\mathfrak{E}, \tag{12}$$

$$\mathfrak{E} = \Phi\mathfrak{F}, \tag{13}$$

$$\Psi = \Phi^{-1}. \tag{14}$$

In other words, the displacements in either theory are subject to the same general and surface conditions as the forces required to maintain the vibrations in an element of volume in the other theory.

To fix our ideas in regard to the signification of Ψ and Φ, we may consider the case of isotropic media, in which these operators reduce to ordinary algebraic quantities, simple or complex. Now the *curl* of any vector necessarily satisfies the solenoidal condition (the so-called "equation of continuity"), therefore by (6) $\Psi\mathfrak{E}$ and $\mathfrak{E}$ will be solenoidal. So also will $\mathfrak{F}$ and $\Phi\mathfrak{F}$ in the electrical theory. Now for solenoidal vectors

$$-\text{curl curl}=\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}, \tag{15}$$

so that the equations (6) and (10) reduce to

$$\Psi\ddot{\mathfrak{E}}=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\mathfrak{E}, \tag{16}$$

$$\ddot{\mathfrak{F}}=\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)\Phi\mathfrak{F}. \tag{17}$$

For a simple train of waves, the displacement, in either theory, may be represented by a constant multiplied by

$$e^{\iota(gt+ax+by+c)} \tag{18}$$

Our equations then reduce again to

$$g^2\Psi\mathfrak{E}=(a^2+b^2+c^2)\mathfrak{E}, \tag{19}$$

$$g^2\mathfrak{F}=(a^2+b^2+c^2)\Phi\mathfrak{F}. \tag{20}$$

Hence

$$\Psi^{-1}=\Phi=\frac{g^2}{a^2+b^2+c^2}. \tag{21}$$

The last member of this equation, when real, evidently expresses the square of the velocity of light. If we set

$$n^2=k^2\frac{a^2+b^2+c^2}{g^2}, \tag{22}$$

k denoting the velocity of light *in vacuo*, we have

$$n^2=k^2\Psi=k^2\Phi^{-1}. \tag{23}$$

When n^2 is positive, which is the case of perfectly transparent bodies, the positive root of n^2 is called the index of refraction of the medium. In the most general case, it would be appropriate to call n—or perhaps that root of n^2 of which the real part is positive—the (complex) index of refraction, although the terminology is hardly settled in this respect. A negative value of n^2 would represent a body from which light would be totally reflected at all angles of incidence. No such cases have been observed. Values of n^2 in which the coefficient of ι is negative, indicate media in which light is

absorbed. Values in which the coefficient of ι is positive would represent media in which the opposite phenomenon took place.*

It is no part of the object of this paper to go into the details by which we may derive, so far as observable phenomena are concerned, Fresnel's law of double refraction for transparent bodies, as well as the more general law of the same character which relates to aeolotropic bodies of more or less opacity, and which differs from Fresnel's only in that certain quantities become complex, or Fresnel's laws for the intensities of reflected and refracted light at the boundary of transparent isotropic media, with the more general laws for the case of bodies aeolotropic or opaque or both. The principal cases have already been discussed on the new elastic theory in the *Philosophical Magazine* † and a further discussion is promised. For the electrical theory, the case of double refraction in perfectly transparent media has been discussed quite in detail in this Journal, ‡ and the intensities of reflected and refracted light have been abundantly deduced from the above conditions by various authors. § So far as all these laws are concerned, the object of this paper will be attained if if it has been made clear that the two theories, in their extreme cases, give identical results. The greater or less degree of elegance, or completeness, or perspicuity, with which these laws may be developed by different authors, should weigh nothing in favor of either theory.

The non-magnetic rotation of the plane of polarization, with the allied phenomena in aeolotropic bodies, lie in a certain sense outside of the above laws, as depending on minute quantities which have been neglected in this discussion. The manner in which these minute quantities affect the equations of motion on the electrical theory has been shown in a former paper, ‖ where these phenomena in transparent bodies are treated quite at length. For the new theory, a discussion of this subject is promised by Mr. Glazebrook.

But the magnetic rotation of the plane of polarization, with the allied phenomena when an aeolotropic body is subjected to magnetic influence, fall entirely within the scope of the above equations and surface-conditions. The characteristic of this case is that Ψ and Φ are not self-conjugate. ¶ This is what we might expect on the electric

* But ι might have been introduced into the equations in such a way that a positive coefficient in the value of n^2 would indicate absorption, and a negative coefficient the impossible case.

† Sir William Thomson, *loc. citat.* R. T. Glazebrook, *loc. citat.*

‡ This vol. p. 182.

§ Lorentz, Schlömilch's *Zeitschrift*, vol. xxii, pp. 1-30 and 205-219; vol. xxiii, pp. 197-210; Fitzgerald, *Phil. Trans.*, vol. clxxi, p. 691; J. J. Thomson, *Phil. Mag.* (5), vol. ix, p. 284; Rayleigh, *Phil. Mag.* (5), vol. xii, p. 81. Glazebrook, *Proc. Cambr. Phil. Soc.*, vol. iv, p. 155.

‖ This vol. p. 195.

¶ See this vol. p. 217.

theory from the experiments of Dr. Hall, which show that the operators expressing the relation between electromotive force and current are not in general self-conjugate in this case.

In the preceding comparison, we have considered only the limiting cases of the two theories. With respect to the sense in which the limiting case is admissible, the two theories do not stand on quite the same footing. In the electric theory, or in any in which the velocity of the missing wave is very great, if we are satisfied that the compressibility is so small as to produce no appreciable results, we may set it equal to zero in our mathematical theory, even if we do not regard this as expressing the actual facts with absolute accuracy. But the case is not so simple with an elastic theory in which the forces resisting certain kinds of motion vanish, so far at least as they are proportional to the strains. The first requisite for any sort of optical theory is that the forces shall be proportional to the displacements. This is easily obtained in general by supposing the displacements very small. But if the resistance to one kind of distortion vanishes, there will be a tendency for this kind of distortion to appear at some places in an exaggerated form, and even to an infinite degree, however small the displacements may be in other parts of the field. In the case before us, if we suppose the velocity of the missing wave to be absolutely zero, there will be infinite condensations and rarefactions at a surface where ordinary waves are reflected. That is, a certain volume of ether will be condensed to a surface, and *vice versâ*. This prevents any treatment of the extreme case, which is at once simple and satisfactory. The difficulty has been noticed by Sir William Thomson, who observes that it may be avoided if we suppose the displacements infinitely small in comparison with the wave-length of the wave of compression. This implies a finite velocity for that wave. A similar difficulty would probably be found to exist (in the extreme case) with regard to the deformation of the ether by the molecules of ponderable matter, as the ether oscillates among them. If the statical resistance to irrotational motions is zero, it is not at all evident that the statical forces evoked by the disturbance caused by the molecules would be proportional to the motions. But this difficulty would be obviated by the same hypothesis as the first.

These circumstances render the elastic theory somewhat less convenient as a working hypothesis than the electric. They do not necessarily involve any complication of the equations of optics. For it may still be possible that this velocity of the missing wave is so small that the quantities on which it depends may be set equal to zero in the equations which represent the phenomena of optics.

But the mental processes by which we satisfy ourselves of the validity of our results (if we do not work out the whole problem in the general case of no assumption in regard to the velocity of the missing wave) certainly involve conceptions of a higher degree of difficulty on account of the circumstances mentioned. Perhaps this ought not to affect our judgment with respect to the question of the truth of the hypothesis.

Although the two theories give laws of exactly the same form for monochromatic light in the limiting case, their deviations from this limit are in opposite directions, so that if the phenomena of optics differed in any marked degree from what we would have in the limiting case, it would be easy to find an *experimentum crucis* to decide between the two theories. A little consideration will make it evident, that when the principal indices of refraction of a crystal are given, the intermediate values for oblique wave-planes will be less if the velocity of the missing wave is small but finite, than if it is infinitesimal, and will be greater if the velocity of the missing wave is very great than if it is infinite.* Hence, if the velocity of the missing wave is small but finite, the intermediate values of the indices of refraction will be less than are given by Fresnel's law, but if the velocity of the missing wave is very great but finite, the intermediate values of the indices of refraction will be greater than are given by Fresnel's law. But the recent experiments of Professor Hastings on the law of double refraction in Iceland spar do not encourage us to look in this direction for the decision of the question.†

In a simple train of waves in a transparent medium, the potential energy, on the elastic theory, may be divided into two parts, of which one is due to that general deformation of the ether which is represented by the equations of wave-motion, and the other to those deformations which are caused by the interference of the ponderable particles with the wave-motion, and to such displacements of the ponderable matter as may be caused, in some cases at least, by the motion of the ether. If we write h for the amplitude, l for the wave-length, and p for the period, these two parts of the statical energy (estimated per unit of volume for a space including many wave-lengths) may be represented respectively by

$$\frac{\pi^2 B h^2}{l^2} \text{ and } \frac{b h^2}{4}.$$

* This may be more clear if we consider the stationary waves formed by two trains of waves moving in opposite directions. The case then comes under the following theorem:

"If the system undergo such a change that the potential energy of a given configuration is diminished, while the kinetic energy of a given motion is unaltered, the periods of the free vibrations are all increased, and conversely." See Lord Rayleigh's *Theory of Sound*, vol. i, p. 85.

† *Am. Jour. Sci.*, ser 3, vol. xxxv, p. 60.

The sum of these may be equated to the kinetic energy, giving an equation of the form

$$\frac{\pi^2 \mathrm{B} h^2}{l^2} + \frac{b h^2}{4} = \frac{\pi^2 \mathrm{A}' h^2}{p^2}. \tag{24}$$

B is an absolute constant (the rigidity of the ether, previously represented by the same letter), A′ and b will be constant (for the same medium and the same direction of the wave-normal) except so far as the type of the motion changes, i.e., except so far as the manner in which the motion of the ether distributes itself between the ponderable molecules, and the degree in which these take part in the motion, may undergo a change. When the period of vibration varies, the type of motion will vary more or less, and A′ and b will vary more or less.

In a manner entirely analogous,* the *kinetic* energy, on the electrical theory, may be divided into two parts, of which one is due to those general fluxes which are represented by the equations of wave-motions, and the other to those irregularities in the fluxes which are caused by the presence of the ponderable molecules, as well as to such motions of the ponderable particles themselves as may sometimes occur. These parts of the kinetic energy may be represented respectively by

$$\frac{\pi \mathrm{F} l^2 h^2}{p^2} \quad \text{and} \quad \frac{\pi^2 f h^2}{p^2}.$$

Their sum equated to the potential energy gives

$$\frac{\pi \mathrm{F} l^2 h^2}{p^2} + \frac{\pi^2 f h^2}{p^2} = \frac{\mathrm{G} h^2}{4}. \tag{25}$$

Here F is the constant of electrodynamic induction, which is unity if we use the electromagnetic system of units, f and G (like A′ and b) vary only so far as the type of motion varies.

We have the means of forming a very exact numerical estimate of the ratio of the two parts into which the statical energy is thus divided on the elastic theory, or the kinetic energy on the electric theory. The means for this estimate is afforded by the principle that the period of a natural vibration is stationary when its type is infinitesimally altered by any constraint.† Let us consider a case of simple wave-motion, and suppose the period to be infinitesimally varied: the wave-length will also vary, and presumably to some extent the type of vibration. But, by the principle just stated, if the ether or the electricity could be constrained to vibrate in the original type, the variations of l and p would be the same as in the actual

* See page 182 of this volume.

† See Lord Rayleigh's *Theory of Sound*, vol. i, p. 84. The application of the principle is most simple in the case of stationary waves.

case. Therefore, in finding the differential equation between l and p, we may treat b and A′ in (24) and f and G in (25) as constant, as well as B and F. These equations may be written

$$4\pi^2 \mathrm{B}\frac{p^2}{l^2} + bp^2 = 4\pi^2 \mathrm{A}',$$

$$\pi \mathrm{F}\frac{l^2}{p^2} + \frac{\pi^2 f}{p^2} = \tfrac{1}{4}\mathrm{G}.$$

Differentiating, we get

$$4\pi^2 \mathrm{B}\, d\frac{p^2}{l^2} = -b\, d(p^2),$$

$$\pi \mathrm{F}\, d\frac{l^2}{p^2} = -\pi^2 f\, d(p^{-2});$$

or

$$4\pi^2 \mathrm{B}\frac{p^2}{l^2}\, d\log\frac{p^2}{l^2} = -bp^2\, d\log p^2.$$

$$\pi \mathrm{F}\frac{l^2}{p^2}\, d\log\frac{l^2}{p^2} = -\frac{\pi^2 f}{p^2}\, d\log p^{-2}.$$

Hence, if we write V for the wave-velocity (l/p), n for the index of refraction, and λ for the wave-length *in vacuo*, we have for the ratio of the two parts into which we have divided the potential energy on the elastic theory,

$$\frac{bh^2}{4} \div \frac{\pi^2 \mathrm{B}h^2}{l^2} = \frac{d\log \mathrm{V}}{d\log p} = -\frac{d\log n}{d\log \lambda}, \tag{26}$$

and for the ratio of the two parts into which we have divided the kinetic energy on the electrical theory,

$$\pi^2 \frac{fh^2}{p^2} \div \frac{\pi \mathrm{F} l^2 h^2}{p^2} = \frac{d\log \mathrm{V}}{d\log p} = -\frac{d\log n}{d\log \lambda}. \tag{27}$$

It is interesting to see that these ratios have the same value. This value may be expressed in another form, which is suggestive of some important relations. If we write U for what Lord Rayleigh has called the velocity of a group of waves,*

$$\frac{\mathrm{U}}{\mathrm{V}} = 1 - \frac{d\log \mathrm{V}}{d\log l},$$

$$\frac{d\log \mathrm{V}}{d\log l} = \frac{\mathrm{V}-\mathrm{U}}{\mathrm{V}},$$

$$\frac{d\log \mathrm{V}}{d\log p} = \frac{\mathrm{V}-\mathrm{U}}{\mathrm{U}}. \tag{28}$$

It appears, therefore, that in the elastic theory that part of the potential energy which depends on the deformation expressed

* See his "Note on Progressive Waves," *Proc. Lond. Math. Soc.*, vol. ix, No. 125, reprinted in his *Theory of Sound*, vol. ii, p. 297.

by the equations of wave-motion, bears to the whole potential energy the same ratio which the velocity of a group of waves bears to the wave-velocity. In the electrical theory, that part of the kinetic energy which depends on the motions expressed by the equations of wave-motion bears to the whole kinetic energy the same ratio.

Returning to the consideration of equations (26) and (27), we observe that in transparent bodies the last member of these equations represents a quantity which is small compared with unity, at least in the visible spectrum, and diminishes rapidly as the wave-length increases. This is just what we should expect of the first member of equation (27). But when we pass to equation (26), which relates to the elastic theory, the case is entirely different. The fact that the kinetic energy is affected by the presence of the ponderable matter and affected differently in different directions, shows that the motion of the ether is considerably modified. This implies a distortion superposed upon the distortion represented by the equations of wave-motion, and very much greater, since the body is very fine-grained as measured by a wave-length. With any other law of elasticity, we should suppose that the energy of this superposed distortion would enormously exceed that of the regular distortion represented by the equations of wave-motion. But it is the peculiarity of this new law of elasticity that there is one kind of distortion, of which the energy is very small, and which is therefore peculiarly likely to occur. Now if we can suppose the distortion caused by the ponderable molecules to be almost entirely of this kind, we may be able to account for the smallness of its energy. We should still expect the first member of (26) to increase with the wave-length, on account of the factor l^2, instead of diminishing, as the last member of the equation shows that it does. We are obliged to suppose that b, and therefore the type of the vibrations, varies very rapidly with the wave-length, even in those cases which appear farthest removed from anything like selective absorption.

The electrical theory furnishes a relation between the refractive power of a body and its specific dielectric capacity, which is commonly expressed by saying that the latter is equal to the square of the index of refraction for waves of infinite length. No objection can be made to this statement, but the great uncertainty in determining the index for waves of infinite length by extrapolation prevents it from furnishing any very rigorous test of the theory. Yet, as the results of extrapolation in some cases agree strikingly with the specific dielectric capacity, although in other cases they are quite different, the correspondence is generally regarded as corroborative, in some degree, of the theory. But the relation between refractive power

and dielectric capacity may be expressed in a form which will furnish a more rigorous test, as not involving extrapolation.

We have seen on page 242 how we may determine numerically the ratio of the two first terms of equation (25). We thus easily get the ratio of the first and last term, which gives

$$\frac{Gh^2}{4}=\frac{d\log l}{d\log\lambda}\frac{\pi F l^2 h^2}{p^2}. \tag{29}$$

In the corresponding equation for a train of waves of the same amplitude and period *in vacuo*, l becomes λ, F remains the same, and for G we may write G′. This gives

$$\frac{G'h^2}{4}=\frac{\pi F\lambda^2 h^2}{p^2}. \tag{30}$$

Dividing, we get

$$\frac{G}{G'}=\frac{d\log l}{d\log\lambda}\frac{l^2}{\lambda^2}=\frac{d(l^2)}{d(\lambda^2)}. \tag{31}$$

Now G′ is the dielectric elasticity of pure ether. If K is the specific dielectric capacity of the body which we are considering, G′/K is the dielectric elasticity of the body and G′/2K is the potential energy of the body (per unit of volume), due to a unit of ordinary electrostatic displacement. But $Gh^2/4$ is the potential energy in a train of waves of amplitude h. Since the average square of the displacement is $h^2/2$, the potential energy of a unit displacement such as occurs in a train of waves is G/2. Now in the electrostatic experiment the displacement distributes itself among the molecules so as to make the energy a minimum. But in the case of light the distribution of the displacement is not determined entirely by statical considerations. Hence

$$\frac{G}{2}\geqq\frac{G'}{2K}, \tag{32}$$

$$K\geqq\frac{G'}{G},$$

and

$$K\geqq\frac{d(\lambda^2)}{d(l^2)}. \tag{33}$$

It is to be observed that if we should assume for a dispersion-formula

$$n^{-2}=a-b\lambda^{-2}, \tag{34}$$

$1/a$, which is the square of the index of refraction for an infinite wave-length, would be identical with the second member of (33).

Another similarity between the electrical and optical properties of bodies consists in the relation between conductivity and opacity. Bodies in which electrical fluxes are attended with absorption of energy absorb likewise the energy of the motions which constitute

light. This is strikingly true of the metals. But the analogy does not stop here. To fix our ideas, let us consider the case of an isotropic body and circularly polarized light, which is geometrically the simplest case although its analytical expression is not so simple as that of plane-polarized light. The displacement at any point may be symbolized by the rotation of a point in a circle. The external force necessary to maintain the displacement $\mathfrak{F}$ is represented by $n^{-2}\mathfrak{F}$. In transparent bodies, for which n^{-2} is a positive number, the force is radial and in the direction of the displacement, being principally employed in counterbalancing the dielectric elasticity, which tends to diminish the displacement. In a conductor n^{-2} becomes complex, which indicates a component of the force in the direction of $\dot{\mathfrak{F}}$, that is, tangential to the circle. This is only the analytical expression of the fact above mentioned. But there is another optical peculiarity of metals, which has caused much remark, viz., that the real part of n^2 (and therefore of n^{-2}) is negative, i.e., the radial component of the force is directed towards the center. This inwardly directed force, which evidently opposes the electrodynamic induction of the irregular part of the motion, is small compared with the outward force which is found in transparent bodies, but increases rapidly as the period diminishes. We may say, therefore, that metals exhibit a second optical peculiarity,—that the dielectric elasticity is not prominent as in transparent bodies. This is like the electrical behavior of the metals, in which we do not observe any elastic resistance to the motion of electricity. We see, therefore, that the complex indices of metals, both in the real and the imaginary part of their inverse squares, exhibit properties corresponding to the electrical behavior of the metals.

The case is quite different in the elastic theory. Here the force from outside necessary to maintain in any element of volume the displacement $\mathfrak{E}$ is represented by $n^2\mathfrak{E}$. In transparent bodies, therefore, it is directed toward the center. In metals, there is a component in the direction of the motion $\dot{\mathfrak{E}}$, while the radial part of the force changes its direction and is often many times greater than the opposite force in transparent bodies. This indicates that in metals the displacement of the ether is resisted by a strong elastic force, quite enormous compared to anything of the kind in transparent bodies, where it indeed exists, but is so small that it has been neglected by most writers except when treating of dispersion. We can make these suppositions, but they do not correspond to anything which we know independently of optical experiment.

It is evident that the electrical theory of light has a serious rival, in a sense in which, perhaps, one did not exist before the publication

of Sir William Thomson's paper in November last.* Nevertheless, neither surprise at the results which have been achieved, nor admiration for that happy audacity of genius, which, seeking the solution of the problem precisely where no one else would have ventured to look for it, has turned half a century of defeat into victory, should blind us to the actual state of the question.

It may still be said for the electrical theory, that it is not obliged to invent hypotheses,† but only to apply the laws furnished by the science of electricity, and that it is difficult to account for the coincidences between the electrical and optical properties of media, unless we regard the motions of light as electrical. But if the electrical character of light is conceded, the optical problem is very different from anything which existed in the time of Fresnel, Cauchy, and Green. The third wave, for example, is no longer something to be gotten rid of *quocunque modo*, but something which we must dispose of in accordance with the laws of electricity. This would seem to rule out the possibility of a relatively small velocity for the third wave.

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* "Since the first publication of Cauchy's work on the subject in 1830, and of Green's in 1837, many attempts have been made by many workers to find a dynamical foundation by Fresnel's laws of reflexion and refraction of light, but all hitherto ineffectually." Sir William Thomson, *loc. citat.*

"So far as I am aware, the electric theory of Maxwell is the only one satisfying these conditions (of explaining at once Fresnel's laws of double refraction in crystals and those governing the intensity of reflexion when light passes from one isotropic medium to another)." Lord Rayleigh, *Phil. Mag.*, September, 1888.

† Electrical motions in air, since the recent experiments of Professor Hertz, seem to be no longer a matter of hypothesis. We can hardly suppose that the case is essentially different with the so-called vacuum. The theorem that the electrical motions of light are solenoidal, although it is convenient to assume it as a hypothesis and show that the results agree with experiment, need not occupy any such fundamental position in the theory. It is in fact only another way of saying that two of the constants of electrical science have a certain ratio (infinity). It would be easy to commence without assuming this value, and to show in the course of the development of the subject that experiment requires it, not of course as an abstract proposition, but in the sense in which experiment can be said to require any values of any constants, that is, to a certain degree of approximation.

XVI.

REVIEWS OF NEWCOMB AND MICHELSON'S "VELOCITY OF LIGHT IN AIR AND REFRACTING MEDIA" AND OF KETTELER'S "THEORETISCHE OPTIK."

[*American Journal of Science*, ser. 3, vol. XXXI. pp. 62-67, Jan. 1886.]

Velocity of Light in Air and Refracting Media.

Astrononomical Papers prepared for the use of the American Ephemeris and Nautical Almanac, vol. II. parts 3 and 4,* Washington, 1885.

PROFESSOR NEWCOMB obtains as the final result of his experiments at Washington 299,860±30 kilometers per second for the velocity of light *in vacuo*. Professor MICHELSON'S entirely independent experiments at Cleveland give substantially the same result (299,853±60) His former experiments at the Naval Academy, after correction of two small errors which he now reports, give 299,910±50. All these experiments were made with the revolving mirror, but the arrangements of the two experimenters were in other respects radically different. The first of these values of the velocity of light with Nyrén's value of the constant of aberration (20″·492) gives 149·60 for the distance of the sun in millions of kilometers. On acount of the recent announcement by Messrs. Young and Forbes of a difference of about two per cent. in the velocities of red and blue light, especial attention was paid to this point by both experimenters, without finding the least indication of any difference. In Professor Newcomb's experiments, a difference of only one thousandth in these velocities would have produced a well-marked iridescence on the edges of the return image of the slit formed by reflection from the revolving mirror. No trace of such iridescence could ever be seen. Professor Michelson made an experiment, in which a red glass covered one-half the slit. The two halves of the image—the upper white, the lower red—were exactly in line.

Since Maxwell's electromagnetic theory of light makes the velocity of light in air equal to the ratio of the electromagnetic and

*[Part 3, "Measures of the Velocity of Light," S. Newcomb; part 4, "Supplementary Measures of the Velocities of white and colored Light in air, water, and carbon disulphide," A. A. Michelson.]

electrostatic units of electricity, it will be interesting to compare some recent determinations of this ratio. These we give in the following table. Since the determinations are affected by any error in the standard of resistance, we have corrected the results, first, on the supposition that the B.A. ohm = ·987 true ohms (Lord Rayleigh's result), and secondly, on the supposition that the B.A. ohm = ·989 true ohms, which is essentially assuming that the legal ohm represents the true value.

Ratio of Electromagnetic and Electrostatic units of Electricity in millions of meters and seconds.

	Date.	As published.	B.A. ohm = ·987.	B.A. ohm = ·989.
Ayrton & Perry,*	1878	298·0	296·1	296·4
Hockin, †	1879	298·8	296·9	297·2
Shida, ‡	1880	299·5	295·6	296·2
Exner, §	1882	301·1 (?)	291·7 (?)	292·3 (?)
J. J. Thomson, ‖	1883	296·3	296·3	296·9
Klemenčič, ¶	1884	301·88 (?)	301·88 (?)	302·48 (?)

These numbers are to be compared with the velocity of light in air, in millions of meters per second, for which Professor Newcomb gives 299·778. Of the electrical determinations, that of J. J. Thomson appears by far the most worthy of confidence. That of Klemenčič—the only one as great as the velocity of light—was obtained by the use of a condenser with glass,—a method which would presumably give too great a ratio. Exner's value is obtained from the mean of three determinations, one of which differed from the others by about three per cent. If we reject this discordant determination, the mean of the other two would give when corrected for resistance 294·4 and 295·0. If we set aside the determinations of Exner and Klemenčič, the remaining four, which represent three different methods, are very accordant, the mean being nearly identical with the result of J. J. Thomson, and about one per cent. less than the velocity of light.

Professor Michelson's experiments on the velocity of light in carbon disulphide afford an interesting illustration of the difference between the velocity of waves and the velocity of groups of waves—a subject which is treated at length in an appendix to the second volume of Lord Rayleigh's *Theory of Sound*. If we write V for the velocity of waves, U for that of a group of waves, L for the wave-length, and T for the period of vibration,

$$V = \frac{L}{T}, \qquad U = \frac{d(T^{-1})}{d(L^{-1})}.$$

For purposes of numerical calculation, it will be convenient to transform these formulæ by the use of λ for the wave-length *in*

* *Phil. Mag.*, (5), vol. vii, p. 277. † *Report Brit. Assoc.*, 1879, p. 285.
‡ *Phil. Mag.*, (5), vol. x, p. 431. § *Sitzungsberichte Wien. Akad.*, vol. lxxxvi, p. 106.
‖ *Phil. Trans.*, vol. clxxiv, p. 707. ¶ *Sitzungsberichte Wien. Akad.*, vol. lxxxix, p. 298.

vacuo, n for the index of refraction of the medium considered, and k for the velocity of light *in vacuo*, which we shall regard as constant, in accordance with general usage. By substitution of these letters we easily obtain

$$\frac{k}{V}=n, \qquad \frac{k}{U}=\frac{d(n\lambda^{-1})}{d(\lambda^{-1})}.$$

The data for the calculation of these quantities for carbon disulphide are given by Verdet (*Annales de Chimie et de Physique*, (3), vol. lxix, p. 470). They give

for the line D, $k/V=1{\cdot}624$, $k/U=1{\cdot}722$,
for the line E, $k/V=1{\cdot}637$, $k/U=1{\cdot}767$.

The quotient of the velocity *in vacuo* divided by the velocity in carbon disulphide, according to Professor Michelson's experiments with the light of an arc lamp, is $1{\cdot}76\pm{\cdot}02$, which agrees very well with k/U. Another theory, which would make the velocity observed in such experiments V^2/U (*Nature*, vol. xxv, p. 52), receives no countenance from these experiments. The value of kU/V^2 would be about 1·53. Some may think that the experiments on water point in a different direction. Taking our data from Beer's *Einleitung in die höhere Optik*, 1853, p. 411, we get

for D, $k/V=1{\cdot}334$, $k/U=1{\cdot}352$, $kU/V^2=1{\cdot}316$,
for E, $k/V=1{\cdot}336$, $k/U=1{\cdot}359$, $kU/V^2=1{\cdot}313$.

The number obtained by experiment was 1,330, which agrees better with k/V, or even with kU/V^2, than with k/U, but the differences are here too small to have much significance.

Theoretische Optik, gegründet auf das Bessel-Sellmeier'sche Princip, zugleich mit den experimentellen Belegen.

Von Dr. E. KETTELER, Professor an der Universität in Bonn. Viewig und Sohn. Braunschweig, 1885.

The principle of Sellmeier, here referred to, relates to vibrations of ponderable particles excited by the etherial vibrations of light, and to the reaction of the former upon the latter. The name of Bessel is added on account of his previous solution of a somewhat analogous problem relating to the pendulum. The object of this work is "to treat theoretical optics in a complete and uniform manner on the new foundation of the simultaneous vibration of etherial and ponderable particles, and to substitute a consistent and systematic new structure for the present conglomerate of more or less disconnected principles." Such a work demands a critical examination, which should not be

undertaken from any narrow point of view. Any faults of detail will be readily forgiven, if the author shall give the theory of optics the πού στῶ which it has sought so long in vain. We may add that if this effort shall not be judged successful by the scientific world, the author will at least have the satisfaction of being associated in his failure with many of the most distinguished names in mathematical physics.

We have sought to test the proposed theory with respect to that law of optics which seems most conspicuous in its definite mathematical form, and in the rigor of the experimental verifications to which it has been subjected, as well as in the magnificent developments to which it has given rise: the law of double refraction due to Huyghens and Fresnel, and geometrically illustrated by the wave-surface of the latter. We cannot find that the law of Fresnel is proved at all in this treatise. We find on the contrary, that a law is deduced which is different from Fresnel's, and inconsistent with it. We do not refer to anything relating to the direction of vibration of the rays in a crystal, which is a point not touched by the experimental verifications to which we have alluded. We shall confine our comparison to those equations from which the direction of vibration has been eliminated, and which therefore represent relations subject to experimental control. For this purpose equation (13) on page 299 is suitable. It reads

$$\frac{u^2}{n_x^2-n^2}+\frac{v^2}{n_y^2-n^2}+\frac{w^2}{n_z^2-n^2}=0,$$

n_x, n_y, n_z being the principal indices of refraction. This the author calls the equation of the wave-surface or surface of ray-velocities. It has the form of the equation of Fresnel's wave-surface, expressed in terms of the direction-cosines and reciprocal of the radius vector, and if u, v, w are the direction-cosines of the *ray*, and n the velocity of light *in vacuo* divided by the so-called *ray-velocity* in the crystal the equation will express Fresnel's law. But it is impossible to give these meanings to u, v, w and n. They are introduced into the discussion in the expression for the vibrations (p. 295), viz.,

$$\rho=\mathfrak{A}\cos 2\pi\left(\frac{t}{\mathrm{T}}-\frac{n(ux+vy+wz)}{\lambda}\right).$$

The form of this equation shows that u, v, w are proportional to the direction-cosines of the *wave-normal*, and as the relation $u^2+v^2+w^2=1$ is afterwards used, they must be the direction-cosines of the wave-normal. *They cannot possibly denote the direction-cosines of the ray*, except in the particular case in which the ray and wave-normal coincide. Again, from the form of this equation, λ/n must be the wave-length in the crystal, and if λ here as elsewhere in the book

(see p. 25) denotes the wave-length *in vacuo* of light of the period considered, which we doubt not is the intention of the author, n must be the wave-length *in vacuo* divided by the wave-length in the crystal, i.e., the velocity of light *in vacuo* divided by the *wave-velocity* in the crystal. With these definitions of u, v, w, and n, equation (13) expresses a law which is different from Fresnel's. Applied to the simple case of a uniaxial crystal, it makes the relation between the wave-velocity of the extraordinary ray and the angle of the wave-normal with the principal axis the same as that of the radius vector and the angle in an ellipse. The law of Huyghens and Fresnel makes the *reciprocal* of the wave-velocity stand in this relation.

The law which our author has deduced has come up again and again in the history of theoretical optics. Professor Stokes (*Report of the British Assoc.*, 1862, part i, p. 269) and Lord Rayleigh (*Phil. Mag.*, (4), vol. xli, p. 525) have both raised the question whether Huyghens and Fresnel might not have been wrong, and it might not be the wave-velocity and not its reciprocal which is represented by the radius vector in an ellipse. The difference is not very great, for if we lay off on the radii vectores of an ellipse distances inversely proportional to their lengths, the resultant figure will have an oval form approaching that of an ellipse when the eccentricity of the original ellipse is small. Rankine appears to have thought that the difference might be neglected (see *Phil. Mag.*, (4), vol. i, pp. 444, 445) at least he claims that his theory leads to Fresnel's law, while really it would give the same law which our author has found. (Concerning Rankine's "splendid failure," and the whole history of the subject, see Sir Wm. Thomson's *Lectures on Molecular Dynamics at the Johns Hopkins University*, chap. xx.) Professor Stokes undertook experiments to decide the question. His result, corroborated by Glazebrook (*Pro. Roy. Soc.*, vol. xx, p. 443; *Phil. Trans.*, vol. clxxi, p. 421), was that Huyghens and Fresnel were right and that the other law was wrong.

To return to our author, we have no doubt from the context that he regards u, v, w, and n as relating to the ray and not to the wave-normal. We suppose that that is the meaning of his remark that the expression for the vibrations (quoted above) is to be referred to the direction of the ray. It seems rather hard not to allow a writer the privilege of defining his own terms. Yet the reader will admit that when the vibrations have been expressed in the above form an inexorable necessity fixes the significance of the direction determined by u, v, w, and leaves nothing in that respect to the choice of the author.

The historical sketches of the development of ideas in the theory of optics, enriched by very numerous references, will be useful to

the student. An exception, however, must be made with respect to the statements concerning the electromagnetic theory of light. We are told (p. 450) that the English theory, founded by Maxwell and represented by Glazebrook and Fitzgerald, makes the plane of polarization coincide with the plane of vibration, while Lorentz, on the basis of Helmholtz's equations comes to the conclusion that these planes are at right angles. Since all these writers make the electrical displacement perpendicular to the plane of polarization, we can only attribute this statement to some confusion between the electrical displacement and the magnetic force or "displacement" at right angles to it. We are also told that Glazebrook's "surface-conditions" which determine the intensity of reflected and refracted light are different from those of Lorentz,—a singular error in view of the fact that Mr. Glazebrook (*Proc. Camb. Phil. Soc.*, vol. iv, p. 166) expressly states that his results are the same as those of Lorentz, Fitzgerald, and J. J. Thomson. We have spent much fruitless labor in trying to discover where and how the expressions were obtained which are attributed to Glazebrook, but in which the notation has been altered. They ought to come from Glazebrook's equations (24)–(27) (*loc. cit.*), but these appear identical with Lorentz's equations (58)–(61) (*Zeitschrift f. Math. u. Phys.*, vol. xxii, p. 27). They might be obtained by interchanging the expressions for vibrations in the plane of incidence and at right angles to it, with two changes of sign.

The reader must be especially cautioned concerning the statements and implications of what has not been done in the electromagnetic theory. These are such as to suggest the question whether the author has taken the trouble to read the titles of the papers which have been published. We refer especially to what is said on pages 248, 249 concerning absorption, dispersion, and the magnetic rotation of the plane of polarization.

In the Experimental Part, with which the treatise closes, we have a comparison of formulæ with the results of experiments by the author and others. The author has been particularly successful in the formula for dispersion. In the case of quartz (p. 545), the formula (with four constants) represents the results of experiment in a manner entirely satisfactory through the entire range of wave-length from 2·14 to 0·214. Those who may not agree with the author's theoretical views will nevertheless be glad to see the results of experiment brought together, and, so far as may be, represented by formulæ.

XVII.

ON THE VELOCITY OF LIGHT AS DETERMINED BY FOUCAULT'S REVOLVING MIRROR.

[*Nature*, vol. XXXIII. p. 582, April 22, 1886.]

It has been shown by Lord Rayleigh and others that the velocity (U) with which a group of waves is propagated in any medium may be calculated by the formula

$$U = V\left(1 - \frac{d\log V}{d\log\lambda}\right),$$

where V is the wave-velocity, and λ the wave-length. It has also been observed by Lord Rayleigh that the fronts of the waves reflected by the revolving mirror in Foucault's experiment are inclined one to another, and in consequence must rotate with an angular velocity

$$\frac{dV}{d\lambda}\alpha,$$

where α is the angle between two successive wave-planes of similar phase. When $dV/d\lambda$ is positive (the usual case), the direction of rotation is such that the following wave-plane rotates towards the position of the preceding (see *Nature*, vol. XXV. p. 52).

But I am not aware that attention has been called to the important fact, that while the individual wave rotates the wave-normal of the group remains unchanged, or, in other words, that if we fix our attention on a point moving with the group, therefore with the velocity U, the successive wave-planes, as they pass through that point, have all the same orientation. This follows immediately from the two formulæ quoted above. For the interval of time between the arrival of two successive wave-planes of similar phase at the moving point is evidently $\lambda/(V-U)$, which reduces by the first formula to $d\lambda/dV$. In this time the second of the wave-planes, having the angular velocity $\alpha dV/d\lambda$, will rotate through an angle α towards the position of the first wave-plane. But α is the angle between the two planes. The second plane, therefore, in passing the moving point, will have exactly the same orientation which the first had. To get a picture of the phenomenon, we may imagine that we are able to see a few inches of the top of a moving carriage-wheel. The individual spokes rotate, while the group maintains a vertical direction.

This consideration greatly simplifies the theory of Foucault's experiment, and makes it evident, I think, that the results of all such experiments depend upon the value of U, and not upon that of V.

The discussion of the experiment by following a single wave, and taking account of its rotation, is a complicated process, and one in which it is very easy to leave out of account some of the elements of the problem. The principal objection to it, however, is its unreality. If the dispersion is considerable, no wave which leaves the revolving mirror will return to it. The individual disappears, only the group has permanence. Prof. Schuster in his communication of March 11 (p. 439), has nevertheless obtained by this method, as the quantity determined by "the experiments hitherto performed," $V^2/(2V-U)$, which, as he observes, is nearly equal to U. He would, I think, have obtained U precisely, if for the angle between two successive wave-planes of similar phase, instead of $2w\lambda/V$, he had used the more exact value $2w\lambda/U$.

By the kindness of Prof. Michelson, I am informed with respect to his recent experiments on the velocity of light in bisulphide of carbon that he would be inclined to place the maximum brilliancy of the light between the spectral lines D and E, but nearer to D. If we take the mean between D and E, we have

$$\frac{K}{U}=1{\cdot}745, \qquad \frac{K(2V-U)}{V^2}=1{\cdot}737,$$

K denoting the velocity *in vacuo* (see p. 249 of this volume). The number observed was 1·76, "with an uncertainty of two units in the second place of decimals." This agrees best with the first formula. The same would be true if we used values nearer to the line D.

J. Willard Gibbs.

New Haven, Connecticut, April 1. [1886.]

XVIII.

VELOCITY OF PROPAGATION OF ELECTROSTATIC FORCE.

[*Nature*, vol. LIII. p. 509, April 2, 1896.]

As we may have to wait some time for the experimental solution of Lord Kelvin's very instructive and suggestive problem concerning two pairs of spheres charged with electricity (see *Nature* of February 6, p. 316), it may be interesting to see what the solution would be from the standpoint of existing electrical theories.

In applying Maxwell's theory to the problem it will be convenient to suppose the dimensions of both pairs of spheres very small in comparison with the unit of length, and the distance between the two pairs very great in comparison with the same unit. These conditions, which greatly simplify the equations which represent the phenomena, will hardly be regarded as affecting the essential nature of the question proposed.

Let us first consider what would happen on the discharge of (A, B), if the system (*c*, *d*) were absent.

Let m_0 be the initial value of the *moment* of the charge of the system (A, B), (this term being used in a sense analogous to that in which we speak of the *moment* of a magnet), and m the value of the moment at any instant. If we set

$$m = \mathrm{F}(t), \tag{1}$$

and suppose the discharge to commence when $t=0$, and to be completed when $t=h$, we shall have

$$\mathrm{F}(t) = m_0 \quad \text{when} \quad t<0, \tag{2}$$

and

$$\mathrm{F}(t) = 0 \quad \text{when} \quad t>h, \tag{3}$$

Let us set the origin of coordinates at the centre of the system (A, B), and the axis of x in the direction of the centre of the positively charged sphere. A unit vector in this direction we shall call i, and the vector from the origin to the point considered ρ. At any point outside of a sphere of unit radius about the origin, the electrical displacement ($\mathfrak{D}$) is given by the vector equation

$$\begin{aligned} 4\pi\mathfrak{D} = {} & [3r^{-5}\mathrm{F}(t-cr)+3cr^{-4}\mathrm{F}'(t-cr)+c^2r^{-3}\mathrm{F}''(t-cr)]x\rho \\ & -[r^{-3}\mathrm{F}(t-cr)+cr^{-2}\mathrm{F}'(t-cr)+c^2r^{-1}\mathrm{F}''(t-cr)]i, \end{aligned} \tag{4}$$

where F denotes the function determined by equation (1), F′ and F″ its derivatives, and c the ratio of electrostatic and electromagnetic

units of electricity, or the reciprocal of the velocity of light. For this satisfies the general equation

$$-\nabla^2\mathfrak{D}=c^2d^2\mathfrak{D}/dt^2, \tag{5}$$

as well as the so-called "equation of continuity," and also satisfies the special conditions that when $t<0$

$$4\pi\mathfrak{D}=m_0(3r^{-5}x\rho-r^{-3}i)$$

outside of the unit sphere, and that at any time at the surface of this sphere

$$4\pi\mathfrak{D}=m(3x\rho-i),$$

if we consider the terms containing the factor c as negligible, when not compensated by large values of r. That equation (4) satisfies the general conditions is easily verified, if we set

$$u=r^{-1}\mathrm{F}(t-cr), \tag{6}$$

and observe that

$$-\nabla^2u=c^2d^2u/dt^2, \tag{7}$$

and that the three components of $\mathfrak{D}$ are given by the equations

$$\left.\begin{aligned} 4\pi f&=-d^2u/dy^2-d^2u/dz^2\\ 4\pi g&=d^2u/dxdy\\ 4\pi h&=d^2u/dxdz \end{aligned}\right\} \tag{8}$$

Equation (4) shows that the changes of the electrical displacement are represented by three systems of spherical waves, of forms determined by the rapidity of the discharge of the system (A, B), which expand with the velocity of light with amplitudes diminishing as r^{-3}, r^{-2}, and r^{-1}, respectively. Outside of these waves, the electrical displacement is unchanged, inside of them it is zero.

If we write (with Maxwell) $-d\mathfrak{A}/dt$ for the force of electrodynamic induction at any point, and suppose its rectangular components calculated from those of $-d^2\mathfrak{D}/dt^2$ by the formula used in calculating the potential of a mass from its density, we shall have by Poisson's theorem

$$\nabla^2(d\mathfrak{A}/dt)=4\pi d^2\mathfrak{D}/dt^2,$$

or by (5),

$$\nabla^2(d\mathfrak{A}/dt)=-4\pi c^{-2}\nabla^2\mathfrak{D},$$

whence

$$d\mathfrak{A}/dt=-4\pi c^{-2}\mathfrak{D}. \tag{9}$$

From this, with (4), and the general equation

$$d\mathfrak{A}/dt+4\pi c^{-2}\mathfrak{D}+\nabla\mathrm{V}=0,$$

we see that during the discharge of the system (A, B) the electrostatic force $-\nabla\mathrm{V}$ vanishes throughout all space, while its place is taken by a precisely equal electrodynamic force $-d\mathfrak{A}/dt$.

This electrodynamic force remains unchanged at every point until the passage of the waves, after which the electrostatic force, the electrodynamic force, and the displacement, have the permanent value zero.

If we write *Curl* for the differentiating vector operator which Maxwell calls by that name, equations (8) may be put in the form

$$4\pi\mathfrak{D} = \text{Curl Curl}\,(iu),$$

whence

$$d\mathfrak{D}/dt = (4\pi)^{-1}\,\text{Curl Curl}\,(i\,du/dt).$$

From $d\mathfrak{D}/dt$ we may calculate the magnetic induction $\mathfrak{B}$ by an operation which is the inverse of $(4\pi)^{-1}$ Curl. We have therefore

$$\mathfrak{B} = \text{Curl}\,(i\,du/dt),$$

or

$$\mathfrak{B} = [r^{-3}\mathrm{F}'(t-cr) + cr^{-2}\mathrm{F}''(t-cr)](yk - zj).$$

The magnetic induction is therefore zero except in the waves.

Equations (4) and (9) give the value of $d\mathfrak{A}/dt$ as function of t and r. By integration, we may find the value of $\mathfrak{A}$, Maxwell's "vector potential." This will be of the form of the second member of (4) multiplied by $-c^{-2}$, if we should give each F one accent less, and for an unaccented F should write F_1, to denote the primitive of F which vanishes for the argument ∞.

That which seems most worthy of notice is that although simultaneously with the discharge of the system (A, B) the values of what we call the electric potential, the electrodynamic force of induction, and the "vector potential," are changed throughout all space, this does not appear connected with any physical change outside of the waves, which advance with the velocity of light.

If we now suppose that there is a second pair of charged spheres (c, d), as in the original problem, the discharge of this pair will evidently occur when the relaxation of electrical displacement reaches it. The time between the discharges is, therefore, by Maxwell's theory, the time required for light to pass from one pair to the other.

It may also be interesting to observe that in the axis of x, on both sides of the origin, $x\rho = r^2 i$, and equation (4) reduces to

$$4\pi\mathfrak{D} = [2r^{-3}\mathrm{F}(t-cr) + 2cr^{-2}\mathrm{F}'(t-cr)]i.$$

Here, therefore, the oscillations are normal to the wave-surfaces. This might seem to imply that plane waves of normal oscillations may be propagated, since we are accustomed to regard a part of an infinite sphere as equivalent to a part of an infinite plane. Of course, such a result would be contrary to Maxwell's theory. The paradox is explained if we consider that the parts of the wave-motion, expressed by F and F′, diminish more rapidly than those expressed by F″, so that it is unsafe to take the displacements in the axis of x as approximately representing those at a moderate distance from it. In fact, if we consider the displacements not merely in the axis of x, but within a cylinder about that axis, and follow the waves to an infinite distance from the origin, we find no approximation to what is usually meant by plane waves with normal oscillations.

J. Willard Gibbs.

New Haven, Conn., March 12 [1896].

XIX.

FOURIER'S SERIES.

[*Nature*, vol. LIX, p. 200, Dec. 29, 1898.]

I SHOULD like to add a few words concerning the subject of Prof. Michelson's letter in *Nature* of October 6. In the only reply which I have seen (*Nature*, October 13), the point of view of Prof. Michelson is hardly considered.

Let us write $f_n(x)$ for the sum of the first n terms of the series

$$\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \text{etc.}$$

I suppose that there is no question concerning the form of the curve defined by any equation of the form

$$y = 2f_n(x).$$

Let us call such a curve C_n. As n increases without limit, the curve approaches a limiting form, which may be thus described. Let a point move from the origin in a straight line at an angle of 45° with the axis of X to the point (π, π), thence vertically in a straight line to the point $(\pi, -\pi)$, thence obliquely in a straight line to the point $(3\pi, \pi)$, etc. The broken line thus described (continued indefinitely forwards and backwards) is the limiting form of the curve as the number of terms increases indefinitely. That is, if any small distance d be first specified, a number n' may be then specified, such that for every value of n greater than n', the distance of any point in C_n from the broken line, and of any point in the broken line from C_n, will be less than the specified distance d.

But this limiting line is not the same as that expressed by the equation

$$y = \operatorname{limit}_{n=\infty} 2f_n(x).$$

The vertical portions of the broken line described above are wanting in the locus expressed by this equation, except the points in which they intersect the axis of X. The process indicated in the last equation is virtually to consider the intersections of C_n with fixed vertical transversals, and seek the limiting positions when n is increased without limit. It is not surprising that this process does not give the vertical portions of the limiting curve. If we should consider the intersections of C_n with horizontal transversals, and

seek the limits which they approach when n is increased indefinitely we should obtain the vertical portions of the limiting curve as well as the oblique portions.

It should be observed that if we take the equation

$$y = 2f_n(x),$$

and proceed to the limit for $n = \infty$, we do not necessarily get $y = 0$ for $x = \pi$. We may get that ratio by first setting $x = \pi$, and then passing to the limit. We may also get $y = 1$, $x = \pi$, by first setting $y = 1$, and then passing to the limit. Now the limit represented by the equation of the broken line described above is not a special or partial limit relating solely to some special method of passing to the limit, but it is the complete limit embracing all sets of values of x and y which can be obtained by any process of passing to the limit.

J. Willard Gibbs.

New Haven, Conn., November 29 [1898].

[*Nature*, vol. lix, p. 606, April 27, 1899.]

I should like to correct a careless error which I made (*Nature*, December 29, 1898) in describing the limiting form of the family of curves represented by the equation

$$y = 2\left(\sin x - \frac{1}{2}\sin 2x \ldots \pm \frac{1}{n}\sin nx\right) \qquad (1)$$

as a zigzag line consisting of alternate inclined and vertical portions. The inclined portions were correctly given, but the vertical portions, which are bisected by the axis of X, extend beyond the points where they meet the inclined portions, their total lengths being expressed by four times the definite integral

$$\int_0^\pi \frac{\sin u}{u}\, du.$$

If we call this combination of inclined and vertical lines C, and the graph of equation (1) C_n, and if any finite distance d be specified, and we take for n any number greater than $100/d^2$, the distance of every point in C_n from C is less than d, and the distance of every point in C from C_n is also less than d. We may therefore call C the limit (or limiting form) of the sequence of curves of which C_n is the general designation.

But this limiting form of the graphs of the functions expressed by the sum (1) is different from the graph of the function expressed by the limit of that sum. In the latter the vertical portions are wanting, except their middle points.

I think this distinction important, for (with exception of what relates to my unfortunate blunder described above) whatever differences of opinion have been expressed on this subject seem due, for the most part, to the fact that some writers have had in mind the *limit of the graphs*, and others the *graph of the limit* of the sum. A misunderstanding on this point is a natural consequence of the usage which allows us to omit the word *limit* in certain connections, as when we speak of the sum of an infinite series. In terms thus abbreviated, either of the things which I have sought to distinguish may be called the graph of the sum of the infinite series.

J. Willard Gibbs.

New Haven, April 12 [1899].

XX.

RUDOLF JULIUS EMANUEL CLAUSIUS.

[*Proceedings of the American Academy*, new series, vol. XVI, pp. 458–465, 1889.]

RUDOLF JULIUS EMANUEL CLAUSIUS was born at Cöslin in Pomerania, January 2, 1822. His studies, after 1840, were pursued at Berlin, where he became Privat-docent in the University, and Instructor in Physics in the School of Artillery. He was Professor of Physics at Zürich in the Polytechnicum (1855–67) and in the University (1857–67), at Würzburg (1867–69), and finally at Bonn (1869–88), where he died on the 24th of August, 1888.

His literary activity commenced in 1847, with the publication of a memoir in *Crelle's Journal*, "Ueber die Lichtzerstreuung in der Atmosphäre, und über die Intensität des durch die Atmosphäre reflectirten Sonnenlichts."* This was immediately followed by other writings relating to the same subject, two of which were subsequently translated from *Poggendorff's Annalen* † for Taylor's *Scientific Memoirs*. A treatise entitled "Die Lichterscheinungen der Atmosphäre" formed part of Grunert's "Beiträge zur meteorologischen Optik."

An entirely different subject, the elasticity of solids, was discussed in his paper (1849), "Ueber die Veränderungen, welche in den bisher gebräuchlichen Formeln für das Gleichgewicht und die Bewegung fester Körper durch neuere Beobachtungen nothwendig geworden sind." ‡

But it was with questions of quite another order of magnitude that his name was destined to be associated. The fundamental questions concerning the relation of heat to mechanical effect, which had been raised by Rumford, Carnot, and others, to meet with little response, were now everywhere pressing to the front.

"For more than twelve years," said Regnault in 1853, "I have been engaged in collecting the materials for the solution of this question:—Given a certain quantity of heat, what is, theoretically, the amount of mechanical effect which can be obtained by applying the heat to evaporation, or the expansion of elastic fluids, in the various circumstances which can be realised in practice?" § The twenty-first

* Vol. xxxiv, p. 122, and vol. xxxvi, p. 185.
† Vol. lxxvi, pp. 161 and 188.
‡ *Pogg. Ann.*, vol. lxxvi, p. 46 (1849).
§ *Comptes Rendus*, vol. xxxvi, p. 676.

volume of the Memoirs of the Academy of Paris, describing the first part of the magnificent series of researches which the liberality of the French government enabled him to carry out for the solution of this question, was published in 1847. In the same year appeared Helmholtz's celebrated memoir, "Ueber die Erhaltung der Kraft." For some years Joule had been making those experiments which were to associate his name with one of the fundamental laws of thermodynamics and one of the principal constants of nature. In 1849 he made that determination of the mechanical equivalent of heat by the stirring of water which for nearly thirty years remained the unquestioned standard. In 1848 and 1849 Sir William Thomson was engaged in developing the consequences of Carnot's theory of the motive power of heat, while Professor James Thomson in demonstrating the effect of pressure on the freezing point of water by a Carnot's cycle, showed the flexibility and the fruitfulness of a mode of demonstration which was to become canonical in thermodynamics. Meantime Rankine was attacking the problem in his own way, with one of those marvellous creations of the imagination of which it is so difficult to estimate the precise value.

Such was the state of the question when Clausius published his first memoir on thermodynamics: "Ueber die bewegende Kraft der Wärme, und die Gesetze, welche sich daraus für die Wärmelehre selbst ableiten lassen." *

This memoir marks an epoch in the history of physics. If we say, in the words used by Maxwell some years ago, that thermodynamics is "a science with secure foundations, clear definitions, and distinct boundaries," † and ask when those foundations were laid, those definitions fixed, and those boundaries traced, there can be but one answer. Certainly not before the publication of that memoir. The materials indeed existed for such a science, as Clausius showed by constructing it from such materials, substantially, as had for years been the common property of physicists. But truth and error were in a confusing state of mixture. Neither in France, nor in Germany, nor in Great Britain, can we find the answer to the question quoted from Regnault. The case was worse than this, for wrong answers were confidently urged by the highest authorities. That question was completely answered, on its theoretical side, in the memoir of Clausius, and the science of thermodynamics came into existence. And as Maxwell said in 1878, so it might have been said at any time since the publication of that memoir, that the foundations of the science were secure, its definitions clear, and its boundaries distinct.

* Read in the Berlin Academy, February 18, 1850, and published in the March and April numbers of *Poggendorff's Annalen*.

† *Nature*, vol. xvii, p. 257.

The constructive power thus exhibited, this ability to bring order out of confusion, this breadth of view which could apprehend one truth without losing sight of another, this nice discrimination to separate truth from error,—these are qualities which place the possessor in the first rank of scientific men.

In the development of the various consequences of the fundamental propositions of thermodynamics, as applied to all kinds of physical phenomena, Clausius was rivalled, perhaps surpassed, in activity and versatility by Sir William Thomson. His attention, indeed, seems to have been less directed toward the development of the subject in extension, than toward the nature of the molecular phenomena of which the laws of thermodynamics are the sensible expression. He seems to have very early felt the conviction, that behind the second law of thermodynamics, which relates to the heat absorbed or given out by a body, and therefore capable of direct measurement, there was another law of similar form but relating to the quantities of heat (i.e., molecular *vis viva*) absorbed in the performance of work, external or internal.

This may be made more definite, if we express the second law in a mathematical form, as may be done by saying that in any reversible cyclic process which a body may undergo

$$\int \frac{dQ}{t} = 0,$$

where dQ is an elementary portion of the heat imparted to the body, and t the absolute temperature of the body, or the portion of it which receives the heat. Or, without limitation to cyclic processes, we may say that for any reversible infinitesimal change,

$$dQ = t\,dS,$$

where S denotes a certain function of the state of the body, called by Clausius the *entropy*. The element of heat may evidently be divided into two parts, of which one represents the increase of molecular *vis viva* in the body, and the other the work done against forces, either external or internal. If we call these parts dH and dQ_w, we have

$$dQ = dH + dQ_w.$$

Now the proposition of which Clausius felt so strong a conviction was that for reversible cyclic processes

$$\int \frac{dQ_w}{dt} = 0,$$

and that for any reversible infinitesimal change

$$dQ_w = t\,dZ,$$

where Z is another function of the state of the body, which he

called the *disgregation*, and regarded as determined by the positions of the elementary parts of the body without reference to their velocities. In this respect it differed from the entropy. An immediate consequence of these relations is that for any reversible cyclic process

$$\int \frac{dH}{t} = 0,$$

and therefore that H, the molecular *vis viva* of the body, must be a function of the temperature alone. This important result was expressed by Clausius in the following words: "Die Menge der in einem Körper wirklich vorhandenen Wärme ist nur von seiner Temperatur und nicht von der Anordnung seiner Bestandtheile abhängig."

To return to the equation

$$dQ_w = t\,dZ.$$

This expresses that heat tends to increase the disgregation, and that the intensity of this tendency is proportional to the absolute temperature. In the words of Clausius: "Die mechanische Arbeit, welche die Wärme bei irgend einer Anordnungsänderung eines Körpers thun kann, ist proportional der absoluten Temperatur, bei welcher die Aenderung geschieht."

Such in brief and in part were the views advanced by Clausius in 1862, in his memoir, "Ueber die Anwendung des Satzes von der Aequivalenz der Verwandlungen auf die innere Arbeit."* Although they were advanced rather as a hypothesis than as anything for which he could give a formal proof, he seems to have little doubt of their correctness, and his confidence seems to have increased with the course of time.

The substantial correctness of these views cannot now be called in question. The researches especially of Maxwell and Boltzmann have shown that the molecular *vis viva* is proportional to the absolute temperature, and Boltzmann has even been able to determine the precise nature of the functions which Clausius called entropy and disgregation.† But the anticipation, to a certain extent, at so early a period in the history of the subject, of the ultimate form which the theory was to take, shows a remarkable insight, which is by no means to be lightly esteemed on account of the acknowledged want of a rigorous demonstration. The propositions, indeed, as relating to quantities which escape direct measurement, belong to molecular science, and seem to require for their complete and satisfactory demonstration a considerable development of that science. This

* *Pogg. Ann.*, vol. cxvi, p. 73. See also vol. cxxvii, p. 477 (1866).

† *Sitzungsberichte Wien. Akad.*, vol. lxiii, p. 728 (1871).

development naturally commenced with the simplest case involving the characteristic problems of the subject,—the case, namely, of gases.

The origin of the kinetic theory of gases is lost in remote antiquity, and its completion the most sanguine cannot hope to see. But a single generation has seen it advance from the stage of vague surmises to an extensive and well established body of doctrine. This is mainly the work of three men, Clausius, Maxwell, and Boltzmann, of whom Clausius was the earliest in the field, and has been called by Maxwell the principal founder of the science.* We may regard his paper (1857), "Ueber die Art der Bewegung, welche wir Wärme nennen,"† as marking his definite entrance into this field, although many points were incidentally discussed in earlier papers.

This was soon followed by his papers, "Ueber die mittlere Länge der Wege, welche bei der Molecularbewegung gasförmiger Körper von den einzelnen Molecülen zurückgelegt werden,"‡ and "Ueber die Wärmeleitung gasförmiger Körper."§

A very valuable contribution to molecular science is the conception of the *virial*, defined in his paper (1870), "Ueber einen auf die Wärme anwendbaren Satz,"‖ where he shows that in any case of stationary motion the mean *vis viva* of the system is equal to its virial.

In the mean time, Maxwell and Boltzmann had entered the field. Maxwell's first paper, "On the Motions and Collisions of perfectly elastic Spheres,"¶ was characterized by a new manner of proposing the problems of molecular science. Clausius was concerned with the mean values of various quantities which vary enormously in the smallest time or space which we can appreciate. Maxwell occupied himself with the relative frequency of the various values which these quantities have. In this he was followed by Boltzmann. In reading Clausius, we seem to be reading mechanics; in reading Maxwell, and in much of Boltzmann's most valuable work, we seem rather to be reading in the theory of probabilities. There is no doubt that the larger manner in which Maxwell and Boltzmann proposed the problems of molecular science enabled them in some cases to get a more satisfactory and complete answer, even for those questions which do not at first sight seem to require so broad a treatment.

Boltzmann's first work, however (1866), "Ueber die mechanische Bedeutung des zweiten Hauptsatzes der Wärmetheorie,"** was in a line in which no one had preceded him, although he was followed by

* *Nature*, vol. xvii, p. 278.
† *Pogg. Ann.*, vol. c, p. 353 (1857).
‡ *Ibid.*, vol. cv, p. 239 (1858). See also *Wied. Ann.*, vol. x, p. 92.
§ *Ibid.*, vol. cxv, p. 1 (1862).
‖ *Ibid.*, vol. cxli, p. 124. See also *Jubelband*, p. 411.
¶ *Phil. Mag.*, vol. xix, p. 19 (1860).
** *Sitzungsberichte Wien. Akad.* vol. liii, p. 195.

some of the most distinguished names among his contemporaries. Somewhat later (1870) Clausius, whose attention had not been called to Boltzmann's work, wrote his paper, "Ueber die Zurückführung des zweiten Hauptsatzes der mechanischen Wärmetheorie auf allgemeine mechanische Principien."*

The point of departure of these investigations, and others to which they gave rise, is the consideration of the mean values of the force-function and of the *vis viva* of a system in which the motions are periodic, and of the variations of these mean values when the external influences are changed. The theorems developed belong to the same general category as the principle of least action, and the principle or principles known as Hamilton's, which have to do, explicitly or implicitly, with the variations of these mean values.

Among other papers of Clausius on this subject, we may mention the two following: "Ueber einen neuen mechanischen Satz in Bezug auf stationäre Bewegung"† (1873), and "Ueber den Satz vom mittleren Ergal und seine Anwendung auf die Molecularbewegungen der Gase" ‡ (1874).

The first problem of molecular science is to derive from the observed properties of bodies as accurate a notion as possible of their molecular constitution. The knowledge we may gain of their molecular constitution may then be utilized in the search for formulas to represent their observable properties. A most notable achievement in this direction is that of van der Waals, in his celebrated memoir, "On the Continuity of the Gaseous and Liquid States." To this part of the subject belong the following papers of Clausius: "Ueber das Verhalten der Kohlensäure in Bezug auf Druck, Volumen und Temperatur,"§ and "Ueber die theoretische Bestimmung des Dampfdruckes und der Volumina des Dampfes und der Flüssigkeit" (two papers). ‖

Another matter in which Clausius showed his originality and power was the vexed subject of electrodynamics, as treated in his memoir, "Ueber die Ableitung eines neuen electrodynamischen Grundgesetzes."¶ Various points in the theory of electricity in which the principles of thermodynamics or of molecular science were involved, had previously been treated in different papers, of which the earliest appeared in 1852,** while the doctrine of the

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* *Pogg. Ann.*, vol. cxlii, p. 433.

† *Ibid.*, vol. cl, p. 106.

‡ *Ibid.*, *Ergänzungsband* vii, p. 215.

§ *Wied. Ann.*, vol. ix, p. 337 (1880).

‖ *Ibid.*, vol. xiv, p. 279 and p. 692 (1881).

¶ *Crelle's Journal*, vol. lxxxii, p. 85 (1877).

** "Ueber das mechanische Aequivalent einer electrischen Entladung und die dabei stattfindende Erwärmung des Leitungsdrahtes." *Pogg. Ann.*, vol. lxxxvi, p. 337. "Ueber die bei einem stationären electrischen Strome in dem Leiter gethane Arbeit

potential (electrical and gravitational) was treated in a separate book, which appeared in 1859, with the title, "Die Potentialfunction und das Potential, ein Beitrag zur mathematischen Physik." This subsequently went through several editions, in which it was revised and enlarged. All these subjects, with others, were brought together in a single volume, "Die mechanische Behandlung der Electricität," which appeared in 1879, forming the second volume of his "Mechanische Wärmetheorie."* Later papers on electricity related to the principles of electrodynamics,† electrical and magnetic units,‡ and dynamo-electric machines.§

The Royal Society's catalogue of scientific papers, and the excellent indices to the *Annalen der Physik und Chemie*, in which Clausius's work usually appeared, render it unnecessary to enumerate in detail his scientific papers. The list, indeed, would be a long one. The Royal Society's catalogue gives seventy-seven titles for the years 1847–1873. Subsequently twenty-five papers have appeared in the *Annalen* alone, and about half as many others elsewhere.

But such work as that of Clausius is not measured by counting titles or pages. His true monument lies not on the shelves of libraries, but in the thoughts of men, and in the history of more than one science.

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und erzeugte Wärme." *Pogg. Ann.*, vol. lxxxvii, p. 415 (1852). "Ueber die Anwendung der mechanischen Wärmetheorie auf die thermoelectrischen Erscheinungen." *Pogg. Ann.*, vol. xc, p. 513 (1853). "Ueber die Electricitätsleitung in Electrolyten." *Pogg. Ann.*, vol. ci, p. 338 (1857).

* The first volume of this work appeared in 1876, and contained the general theory with the more immediate consequences of the two fundamental laws. The third volume has not yet appeared, but it is expected very soon, edited by Professor Planck and Dr. Pulfrich. In a certain sense this work may be regarded as a second edition of an earlier one (1864 and 1867), which consisted of a reprint of papers and had the title "Abhandlungen über die mechanische Wärmetheorie."

† *Wied. Ann.*, vol. x, p. 608; vol. xi, p. 604.

‡ *Ibid.*, vol. xvi, p. 529; vol. xvii, p. 713.

§ *Ibid.*, vol. xx, p. 353; vol. xxi, p. 385.

XXI.

HUBERT ANSON NEWTON.

[*American Journal of Science*, ser. 4, vol. III, pp. 359–376, May 1897.]

(Read before the National Academy of Sciences, in April, 1897.)

HUBERT ANSON NEWTON was born on March 19th, 1830, at Sherburne, N.Y., and died at New Haven, Conn., on the 12th day of August, 1896. He was the fifth son of a family of seven sons and four daughters, children of William and Lois (Butler) Newton. The parents traced their ancestory back to the first settlers of Massachusetts and Connecticut,* and had migrated from the latter to Sherburne, when many parts of central New York were still a wilderness. They both belonged to families remarkable for longevity, and lived themselves to the ages of ninety-three and ninety-four years. Of the children, all the sons and two daughters were living as recently as the year 1889, the youngest being then fifty-three years of age. William Newton was a man of considerable enterprise, and undertook the construction of the Buffalo section of the Erie canal, as well as other work in canal and railroad construction in New York and Pennsylvania. In these constructions he is said to have relied on his native abilities to think out for himself the solution of problems which are generally a matter of technical training. His wife was remarkable for great strength of character united with a quiet temperament and well-balanced mind, and was noted among her neighbors for her mathematical powers.

Young Newton, whose mental endowments were thus evidently inherited, and whose controlling tastes were manifested at a very early age, fitted for college at the schools of Sherburne, and at the age of sixteen entered Yale College in the class graduating in 1850. After graduation he pursued his mathematical studies at New Haven and at home, and became tutor at Yale in January, 1853, when on account of the sickness and death of Professor Stanley the whole charge of the mathematical department devolved on him from the first.

* Richard Butler, the great-grandfather of Lois Butler, came over from England before 1633, and was one of those who removed from Cambridge to Hartford. An ancestor of William Newton came directly from England to the New Haven colony about the middle of the same century.

In 1855, he was appointed professor of mathematics at the early age of twenty-five. This appointment testifies to the confidence which was felt in his abilities, and is almost the only instance in which the Yale Corporation has conferred the dignity of a full professorship on so young a man.

This appointment being accompanied with a leave of absence for a year, in order to give him the opportunity to study in Europe, it was but natural that he should be attracted to Paris, where Chasles was expounding at the Sorbonne that modern higher geometry of which he was to so large an extent the creator, and which appeals so strongly to the sense of the beautiful. And it was inevitable that the student should be profoundly impressed by the genius of his teacher and by the fruitfulness and elegance of the methods which he was introducing. The effect of this year's study under the inspiring influence of such a master is seen in several contributions to the *Mathematical Monthly* during its brief existence in the years 1858-61. One of these was a problem which attracted at once the attention of Cayley, who sent a solution. Another was a discussion of the problem "to draw a circle tangent to three given circles," remarkable for his use of the principle of inversion. A third was a very elaborate memoir on the construction of curves by the straight edge and compasses, and by the straight edge alone. These early essays in geometry show a mind thoroughly imbued with the spirit of modern geometry, skilful in the use of its methods, and eager to extend the bounds of our knowledge.

Nevertheless, although for many years the higher geometry was with him a favorite subject of instruction for his more advanced students, either his own preferences, or perhaps rather the influence of his environment, was destined to lead him into a very different field of research. In the attention which has been paid to astronomy in this country we may recognize the history of the world repeating itself in a new country in respect to the order of the development of the sciences, or it may be enough to say that the questions which nature forces on us are likely to get more attention in a new country and a bustling age, than those which a reflective mind puts to itself, and that the love of abstract truth which prompts to the construction of a system of doctrine, and the refined taste which is a critic of methods of demonstration, are matters of slow growth. At all events, when Professor Newton was entering upon his professorship, the study of the higher geometry was less consonant with the spirit of the age in this country than the pursuit of astronomical knowledge, and the latter sphere of activity soon engrossed his best efforts.

Yet it was not in any of the beaten paths of astronomers that Professor Newton was to move. It was rather in the wilds of a

terra incognita, which astronomers had hardly troubled themselves to claim as belonging to their domain, that he first labored to establish law and order. It was doubtless not by chance that he turned his attention to the subject of shooting stars. The interest awakened in this country by the stupendous spectacle of 1833, which was not seen in Europe, had not died out. This was especially true at New Haven, where Mr. Edward C. Herrick was distinguished for his indefatigable industry both in personal observation and in the search for records of former showers. A rich accumulation of material was thus awaiting development. In 1861, the Connecticut Academy of Arts and Sciences appointed a committee "to communicate with observers in various localities for combined and systematic observations upon the August and November meteors." In this committee Professor Newton was preeminently active. He entered zealously upon the work of collecting material by personal observation and correspondence and by organizing corps of observers of students and others, and at the same time set himself to utilize the material thus obtained by the most careful study. The value of the observations collected was greatly increased by a map of the heavens for plotting meteor-paths, which was prepared by Professor Newton and printed at the expense of the Connecticut Academy for distribution among observers.

By these organized efforts, in a great number of cases observations were obtained on the same meteor as seen from different places, and the actual path in the atmosphere was computed by Professor Newton. In a paper published in 1865 * the vertical height of the beginning and the end of the visible part of the path is given for more than one hundred meteors observed on the nights of August 10th and November 13th, 1863. It was shown that the average height of the November meteors is fifteen or twenty miles higher than that of the August meteors, the former beginning in the mean at a height of ninety-six miles and ending at sixty-one, the latter beginning at seventy and ending at fifty-six.

We mention this paper first, because it seems to represent the culmination of a line of activity into which Professor Newton had entered much earlier. We must go back to consider other papers which he had published in the meantime.

His first papers on this subject, 1860–62,† were principally devoted to the determination of the paths and velocities of certain brilliant meteors or fireballs, which had attracted the attention of observers in different localities. Three of these appeared to have velocities much greater than is possible for permanent members of the solar system. To another a particular interest attached as belonging to the August

* *Amer. Jour. Sci.*, ser. 2, vol. xl, p. 250.

† *Amer. Jour. Sci.*, ser. 2, vol. xxx, p. 186; xxxii, p. 448; and xxxiii, p. 338.

shower, although exceptional in size. For this he calculated the elements of the orbit which would give the observed path and velocity. But the determination of the velocity in such cases, which depends upon the estimation by the observers of the time of flight, is necessarily very uncertain, and at best affords only a lower limit for the value of the original velocity of the body before it encountered the resistance of the earth's atmosphere. This would seem to constitute an insuperable difficulty in the determination of the orbits of meteoroids, to use the term which Professor Newton applied to these bodies, before they enter the earth's atmosphere to appear for a moment as luminous meteors. Yet it has been completely overcome in the case of the November meteors, or Leonids as they are called from the constellation from which they appear to radiate. This achievement constitutes one of the most interesting chapters in the history of meteoric science, and gives the subject an honorable place among the exact sciences.

In the first place, by a careful study of the records, Professor Newton showed that the connection of early showers with those of 1799 and 1833 had been masked by a progressive change in the time of the year in which the shower occurs. This change had amounted to a full month between A.D. 902, when the shower occurred on October 13, and 1833, when it occurred on November 13. It is in part due to the precession of the equinoxes, and in part to the motion of the node where the earth's orbit meets that of the meteoroids. This motion must be attributed to the perturbations of the orbits of the meteoroids which are produced by the attractions of the planets, and being in the direction opposite to that of the equinoxes, Professor Newton inferred that the motion of the meteoroids must be retrograde.

The showers do not, however, occur whenever the earth passes the node, but only when the passage occurs within a year or two before or after the termination of a cycle of 32·25 years. This number is obtained by dividing the interval between the showers of 902 and 1833 by 28, the number of cycles between these dates, and must therefore be a very close approximation. For if these showers did not mark the precise end of cycles the resultant error would be divided by 28. Professor Newton showed that this value of the cycle requires that the number of revolutions performed by the meteoroids in one year should be either $2 \pm \frac{1}{33 \cdot 25}$ or $1 \pm \frac{1}{33 \cdot 25}$ or $\frac{1}{33 \cdot 25}$. In other words, the periodic time of the meteoroids must be either 180·0 or 185·4 or 354·6 or 376·6 days, or 33·25 years. Now the velocity of any body in the solar system has a simple relation to its periodic time and its distance from the sun. Assuming, therefore, any one of these five values of the periodic time, we have the velocities of the Leonids at the node very sharply determined. From this velocity, with the

position of the apparent radiant, which gives the direction of the relative motion, and with the knowledge that the heliocentric motion is retrograde, we may easily determine the orbit.

We have, therefore, five orbits from which to choose. The calculation of the secular motion of the node due to the disturbing action of the planets, would enable us to decide between these orbits.

Such are the most important conclusions which Professor Newton derived from the study of these remarkable showers, interesting not only from the magnificence of the spectacle occasionally exhibited, but in a much higher degree from the peculiarity in the periodic character of their occurrence, which affords the means of the determination of the orbit of the meteoroids with a precision which would at first sight appear impossible.

Professor Newton anticipated a notable return of the shower in 1866, with some precursors in the years immediately preceding, a prediction which was amply verified. In the meantime he turned his attention to the properties which belong to shooting stars in general, and especially to those average values which relate to large numbers of these bodies not belonging to any particular swarm.

This kind of investigation Maxwell has called *statistical*, and has in more than one passage signalized its difficulties. The writer recollects a passage of Maxwell which was pointed out to him by Professor Newton, in which the author says that serious errors have been made in such inquiries by men whose competency in other branches of mathematics was unquestioned. Doubtless Professor Newton was very conscious of the necessity of caution in these inquiries, as is indeed abundantly evident from the manner in which he expressed his conclusions; but the writer is not aware of any passage in which he has afforded an illustration of Maxwell's remark.

The results of these investigations appeared in an elaborate memoir "On Shooting Stars," which was read to the National Academy in 1864, and appeared two years later in the *Memoirs of the Academy*.* An abstract was given in the *American Journal of Science* in 1865.† The following are some of the subjects treated, with some of the more interesting results:

The distribution of the apparent paths of shooting stars in azimuth and altitude.

The vertical distribution of the luminous part of the real paths. The value found for the mean height of the middle point of the luminous path was a trifle less than sixty miles.

The mean length of apparent paths.

The mean distance of paths from the observer.

* Vol. i, 3d memoir.

† Series 2, vol. xxxix, p. 193.

The mean foreshortening of paths.

The mean length of the visible part of the real paths.

The mean time of flight as estimated by observers.

The distribution of the orbits of meteoroids in the solar system.

The daily number of shooting stars, and the density of the meteoroides in the space which the earth traverses.

The average number of shooting stars which enter the atmosphere daily, and which are large enough to be visible to the naked eye, if the sun, moon and clouds would permit it, is more than seven and a half millions. Certain observations with instruments seem to indicate that this number should be increased to more than four hundred millions, to include telescopic shooting stars, and there is no reason to doubt that an increase of optical power beyond that employed in these observations would reveal still larger numbers of these small bodies. In each volume of the size of the earth, of the space which the earth is traversing in its orbit about the sun, there are as many as thirteen thousand small bodies, each of which is such as would furnish a shooting star visible under favorable circumstances to the naked eye.

These conclusions are certainly of a startling character, but not of greater interest than those relating to the velocity of meteoroids. There are two velocities to be considered, which are evidently connected, the velocity relative to the earth, and the velocity of the meteoroids in the solar system. To the latter, great interest attaches from the fact that it determines the nature of the orbit of the meteoroid. A velocity equal to that of the earth, indicates an orbit like that of the earth; a velocity $\sqrt{2}$ times as great, a parabolic orbit like that of most comets, while a velocity greater than this indicates a hyperbolic orbit.

Professor Newton sought to form an estimate of this critical quantity in more than one way. That on which he placed most reliance was based on a comparison of the numbers of shooting stars seen in the different hours of the night. It is evident that in the morning, when we are in front of the earth in its motion about the sun, we should see more shooting stars than in the evening, when we are behind the earth; but the greater the velocity of the meteoroids compared with that of the earth, the less the difference would be in the numbers of evening and morning stars.*

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* It may not be out of place to notice here an erratum which occurs both in the *Memoirs of the National Academy* and in the abstract in the *American Journal of Science*, and which the writer finds marked in a private copy of Professor Newton's. In the table on p. 20 of the memoir and 206 of the abstract, the column of numbers under the head "hour of the night" should be inverted. There is another displacement in the table in the memoir, which is, however, corrected in the abstract.

After a careful discussion of the evidence Professor Newton reached the conclusion that "we must regard as almost certain (on the hypothesis of an equable distribution of the directions of absolute motions), that the mean velocity of the meteoroids exceeds considerably that of the earth; that the orbits are not approximately circular, but resemble more the orbits of comets."

This last sentence, which is taken from the abstract published in the *American Journal of Science* in 1865, and is a little more definitely and positively expressed than the corresponding passage in the original memoir, indicating apparently that the author's conviction had been growing more positive in the interval, or at least that the importance of the conclusion had been growing upon him, embodies what is perhaps the most important result of the memoir, and derives a curious significance from the discoveries which were to astonish astronomers in the immediate future.

The return of the November or Leonid shower in 1865, and especially in 1866, when the display was very brilliant in Europe, gave an immense stimulus to meteoric study, and an especial prominence to this group of meteoroids. "Not since the year 1759," says Schiaparelli, "when the predicted return of a comet first took place, had the verified prediction of a periodic phenomenon made a greater impression than the magnificent spectacle of November, 1866. The study of cosmic meteors thereby gained the dignity of a science, and took finally an honorable place among the other branches of astronomy."* Professor J. C. Adams, of Cambridge, England, then took up the calculation of the perturbations determining the motion of the node. We have seen that Professor Newton had shown that the periodic time was limited to five sharply determined values, each of which with the other data would give an orbit, and that the true orbit could be distinguished from the other four by the calculation of the secular motion of the node.

Professor Adams first calculated the motion of the node due to the attractions of Jupiter, Venus, and the Earth for the orbit having a period of 354·6 days. This amounted to a little less than 12′ in 33·25 years. As Professor Newton had shown that the dates of the showers require a motion of 29′ in 33·25 years, the period of 354·6 days must be rejected. The case would be nearly the same with a period of 376·6 days, while a period of 180 or 185·4 days would give a still smaller motion of the node. Hence, of the five possible periods indicated by Professor Newton, four were shown to be entirely incompatible with the motion of the node, and it only remained to

* Schiaparelli, *Entwurf einer astronomischen Theorie der Sternschnuppen*, p. 55

examine whether the fifth period, viz. that of 33·25 years, would give a motion of the node in accordance with the observed value. As this period gives a very long ellipse for the orbit, extending a little beyond the orbit of Uranus, it was necessary to take account of the perturbations due to that planet and to Saturn. Professor Adams found 28′ for the motion of the node. As this value must be regarded as sensibly identical with Professor Newton's 29′ of observed motion, no doubt was left in regard to the period of revolution or the orbit of the meteoroids.*

About this time, M. Schiaparelli was led by a course of reasoning similar to Professor Newton's to the same conclusion,—that the mean velocity of the meteoroids is not very different from that due to parabolic orbits. In the course of his speculations in regard to the manner in which such bodies might enter the solar system, the questions suggested themselves: whether meteoroids and comets may not have a similar origin; whether, in case a swarm of meteoroids should include a body of sufficient size, this would not appear as a comet; and whether some of the known comets may not belong to streams of meteoroids. Calculating the orbit of the Perseids, or August meteoroids, from the radiant point, with the assumption of a nearly parabolic velocity, he found an orbit very similar to that of the great comet of 1862, which may therefore be considered as one of the Perseids,—probably the largest of them all.†

At that time no known cometic orbit agreed with that of the Leonids, but a few months later, as soon as the definitive elements of the orbit of the first comet of 1866 were published, their resemblance to those of the Leonids, as calculated for the period of 33·25 years, which had been proved to be the correct value, was strikingly manifested, attracting at once the notice of several astronomers.

Other relations of the same kind have been discovered later, of which that of Biela's comet and the Andromeds is the most interesting, as we have seen the comet breaking up under the influence of the sun; but in no case is the coincidence so striking as in that of the Leonids, since in no other case is the orbit of the meteoroids completely known, independently of that of the comet, and without any arbitrary assumption in regard to their periodic time.

The first comet of 1866 is probably not the only one belonging to the Leonid stream of meteoroids. Professor Newton has remarked that the Chinese annals mention two comets which passed rapidly in succession across the sky in 1366, a few days after the passage of the earth through the node of the Leonid stream, which was marked in Europe by one of the most remarkable star-showers on record. The

* *Monthly Notices Roy. Ast. Soc.*, vol. xxvii, p. 247. † *Entwurf*, etc., pp. 49–54.

course of these comets, as described by the annalists, was in the line of the Leonid stream.*

This identification of comets with meteors or shooting-stars marks an epoch in the study of the latter. Henceforth, they must be studied in connection with comets. It was presumably this discovery which led Professor Newton to those statistical investigations respecting comets, which we shall presently consider. At this point, however, at the close as it were of the first chapter in the history of meteoric science, it seems not unfitting to quote the words of an eminent foreign astronomer, written about this time, in regard to Professor Newton's contributions to this subject. In an elaborate memoir in the *Comptes Rendus*, M. Faye says, with reference to our knowledge of shooting-stars and their orbits, "we may find in the works of M. Newton, of the United States, the most advanced expression of the state of science on this subject, and even the germ, I think, of the very remarkable ideas brought forward in these last days by M. Schiaparelli and M. Le Verrier." †

The first fruit of Professor Newton's statistical studies on comets appeared in 1878 in a paper "On the Origin of Comets." In this paper he considers the distribution in the solar system of the known cometic orbits, and compares it with what we might expect on either of two hypotheses: that of Kant, that the comets were formed in the evolution of the solar system from the more distant portion of the solar nebula; and that of Laplace, that the comets have come from the stellar spaces and in their origin had no relation to the solar system.

In regard to the distribution of the aphelia, he shows that, except so far as modified by the perturbations due to the planets, the theory of internal origin would require all the aphelia to be in the vicinity of the ecliptic; the theory of external origin would make all directions of the aphelia equally probable, i.e., the distribution in latitude of the aphelia should be that in which the frequency is as the cosine of the latitude. The actual distribution comes very near to this, but as the effect of perturbations would tend to equalize the distribution of aphelia in all directions, Professor Newton does not regard this argument as entirely decisive. He remarks, however, that if Kant's hypothesis be true, the comets must have been revolving in their orbits a very long time, and the process of the disintegration of comets must be very slow.

In regard to the distribution of the orbits in inclination, the author shows that the theory of internal origin would make all inclinations equally probable; the theory of external origin would make all

* *Amer. Jour. Sci.*, ser. 2, vol. xliii, p. 298, and vol. xlv, p. 91, or *Encycl. Britann.*, article Meteor.

† *Comptes Rendus*, t. lxiv, p. 551.

directions of the normal to the plane of the orbit equally probable. On the first hypothesis, therefore, we should expect a uniform distribution in inclination; on the second, a frequency proportioned to the sine of the inclination. It was shown by a diagram in which the actual and the two theoretical distributions are represented graphically, that the actual distribution agrees pretty well with the theory of external origin and not at all with that of internal origin. It was also shown that the curve of actual distribution cannot be made to agree with Kant's hypothesis by any simple and reasonable allowances for perturbations. On the other hand, if we assume the external origin of comets, and ask how the curve of sines must be modified in order to take account of perturbations, it is shown that the principal effect will be to increase somewhat the number of inclinations between 90° and 135° at the expense of those between 45° and 90°. It is apparent at once from the diagram that such a change would make a very good agreement between the actual and theoretical curves, the only important difference remaining being due to comets of short periods, which mostly have small inclinations with direct motion. These should not weigh very much, the author observes, in the general question of the distribution of inclinations, because they return so frequently and are so easily detected that their number in a list of observed comets is out of all proportion to their number among existing comets. But this group of comets of short periods can easily be explained on the theory of an external origin. For such comets must have lost a large part of their velocity by the influence of a planet. This is only likely to happen when a comet overtakes the planet and passes in front of it. This implies that its original motion was direct and in an orbit of small inclination to that of the planets, and although it may lose a large part of its velocity, its motion will generally remain direct and in a plane of small inclination. This very interesting case of the comets of short periods and small inclinations, which was treated rather briefly in this paper, was discussed more fully by Professor Newton at the meeting of the British Association in the following year.*

Many years later, Professor Newton returned to the same general subject in a very interesting memoir "On the Capture of Comets by Planets; especially their Capture by Jupiter," which was read before the National Academy in 1891, and appeared in the *Memoirs of the Academy* two years later.† It also appeared in the *American Journal of Science* in the year in which it was read.‡ This contains

* *Rep't Brit. Assoc. Adv. Sci.* for 1879, p. 272.

† *Mem. Nat. Acad.*, vol. vi, 1st memoir.

‡ *Amer. Jour. Sci.*, ser. 3, vol. xlii, pp. 183 and 482.

the results of careful statistical calculations on the effect of perturbations on orbits of comets originally parabolic. It corroborates the more general statements of the paper "On the Origin of Comets," giving them a precise quantitative form. One or two quotations will give some idea of the nature of this very elaborate and curious memoir, in which, however, the results are largely presented in the form of diagrams.

On a certain hypothesis regarding an original equable distribution of comets in parabolic orbits about the sun, it is shown that "if in a given period of time a thousand million comets come in parabolic orbits nearer to the sun than Jupiter, 126 of them will have their orbits changed" by the action of that planet "into ellipses with periodic times less than one-half that of Jupiter; 839 of them will have their orbits changed into ellipses with periodic times less than that of Jupiter; 1701 of them will have their orbits changed into ellipses with periodic times less than once and a half that of Jupiter, and 2670 of them will have their orbits changed into ellipses with periodic times less than twice that of Jupiter." A little later, Professor Newton considers the question, which he characterizes as perhaps more important, of the direct or retrograde motion of the comets after such perturbations. It is shown that of the 839 comets which have periodic times less than Jupiter, 203 will have retrograde motions, and 636 will have direct motions. Of the 203 with retrograde motion, and of the 636 with direct motion, 51 and 257, respectively, will have orbits inclined less than 30° to that of Jupiter.

We have seen that the earliest of Professor Newton's more important studies on meteors related to the Leonids, which at that time far surpassed all other meteoric streams in interest. One of his later studies related to another stream which in the mean time had acquired great importance. The identification of the orbit of the Andromed meteors with that of Biela's comet, which we have already mentioned, gave these bodies a unique interest, as the comet had been seen to break up under the influence of the sun. Here the evolution of meteoroids was taking place before our eyes; and this interest was heightened by the showers of 1872 and 1885, which in Europe seem to have been unsurpassed in brilliancy by any which have occurred in this century.

The phenomena of each of these showers were carefully discussed by Professor Newton. Among the principal results of his paper on the latter shower are the following:*

The time of the maximum frequency of meteors was Nov. 27, 1885 6 h. 15 m. Gr. m. t. The estimated number per hour visible at one

* *Amer. Jour. Sci.*, ser. 3, vol. xxxi, p. 409.

place was then 75,000. This gives a density of the meteoroids in space represented by one to a cube of twenty miles edge. Three hours later the frequency had fallen to one-tenth of the maximum value. The really dense portion of the stream through which we passed was less than 100,000 miles in thickness, and nearly all would be included in a thickness of 200,000 miles.

A formula is given to express the effect of the earth's attraction on the approaching meteoroids in altering the position of the radiant. This is technically known as the zenithal attraction, and is quite important in the case of these meteors on account of their small relative velocity. The significance of the formula may be roughly expressed by saying that the earth's attraction changes the radiant of the Biela meteors, toward the vertical of the observer, one-tenth of the observed zenith distance of the radiant, or more briefly, that the zenithal attraction for these meteors is one-tenth of the observed zenith distance. The radiant even after the correction for zenithal attraction, and another for the rotation of the earth on its axis, is not a point but an area of several degrees diameter. The same has been observed in regard to other showers, but the result comes out more distinctly in the present case because the meteors were so numerous and the shower so carefully observed.

This implies a want of parallelism in the paths of the meteors, and it is a very important question whether it exists before the meteoroids enter our atmosphere, or whether it is due to the action of the atmosphere.

Professor Newton shows that it is difficult to account for so large a difference in the original motions of the meteoroids, and thinks it reasonable to attribute a large part of the want of parallelism to the action of the atmosphere on bodies of an irregular form, such as we have every reason to believe that the meteoroids have, when they enter our atmosphere. The effect of the heat generated will be to round off the edges and prominent parts, and to reduce the meteor to a form more and more spherical. It is, therefore, quite natural that the greater portion of the curvature of the paths should be in the invisible portion and thus escape our notice. It is only in exceptional cases that the visible path is notably curved.

But the great interest of the paper centers in his discussion of the relation of this shower to preceding showers, and to the orbit of Biela's comet. The changes in the date of the shower (from Dec. 6 to Nov. 27) and in the position of the radiant are shown to be related to the great perturbations of Biela's comet in 1794, 1831, and 1841-2. The showers observed by Brandes, Dec. 6th, 1798, by Herrick, Dec. 7th, 1838, and by Heis, Dec. 8th and 10th, 1847, are related to the orbit of Biela's comet as it was in 1772; while the great showers of 1872

and 1885, as well as a trifling display in 1867, are related to the orbit of 1852.*

Assuming, then, that the meteoroids which we met on the 27th of November, 1872, did not leave the immediate neighborhood of the Biela comet before 1841–2, we seem to have the data for a very precise determination of their orbit between those dates. The same is true of those which we met in 1885. The computation of these orbits, the author remarks, may possibly give evidence for or against the existence of a resisting medium in the solar system.

In his last public utterance on the subject of meteors, which was on the occasion of the recent sesquicentennial celebration of the American Philosophical Society, Professor Newton returns to the Biela meteoroids, and finds in the scattering which they show in the plane of their orbit the proof of a disturbing force in that plane, and therefore not due to the planets. The force exerted by the sun appears to be modified somewhat as we see it in the comet's tails, where indeed the attraction is changed into a repulsion. Something of the same sort on a smaller scale relatively to the mass of the bodies appears to modify the sun's action on the meteoroids.

In 1888 Professor Newton read a paper before the National Academy "Upon the relation which the former Orbits of those Meteorites that are in our collections, and that were seen to fall, had to the Earth's Orbit." This was based upon a very careful study of more than 116 cases for which we have statements indicating more or less definitely the direction of the path through the air, as well as 94 cases in which we only know the time of the fall. The results are expressed in the following three propositions:

1. The meteorites which we have in our cabinets and which were seen to fall were originally (as a class, and with a very small number of exceptions) moving about the sun in orbits that had inclinations less than 90°; that is, their motions were direct, not retrograde.

2. The reason why we have only this class of stones in our collections is not one wholly or even mainly dependent on the habits of men; nor on the times when men are out of doors; nor on the places where men live; nor on any other principle of selection acting at or after the arrival of the stones at the ground. Either the stones which are moving in the solar system across the earth's orbit move in general in direct orbits; or else for some reason the stones which

* It is a curious coincidence that the original discoverer of the December shower as a periodic phenomenon, Mr. Edward C. Herrick, should have been (with a companion, Mr. Francis Bradley,) the first to observe that breaking up of the parent body which was destined to reinforce the meteoric stream in so remarkable a manner. See *Amer. Jour. Sci.*, ser. 3, vol. xxxi, pp. 85 and 88.

move in retrograde orbits do not in general come through the air to the ground in solid form.

3. The perihelion distances of nearly all the orbits in which these stones moved were not less than 0·5 nor more than 1·0, the earth's radius vector being unity.

Professor Newton adds, that it seems a natural and proper corollary to these propositions (unless it shall appear that stones meeting the earth are destroyed in the air) that the larger meteorites moving in our solar system are allied much more closely with the group of comets of short period than with comets whose orbits are nearly parabolic. All the known comets of shorter periods than 33 years move about the sun in direct orbits that have moderate inclinations to the ecliptic. On the contrary, of the nearly parabolic orbits that are known only a small proportion of the whole number have small inclinations with direct motion.

We have briefly mentioned those papers which seem to constitute the most important contributions to the science of meteors and comets. To fully appreciate Professor Newton's activity in this field, it would be necessary to take account of his minor contributions.*

Most interesting and instructive to the general reader are his utterances on occasions when he has given a résumé of our knowledge on these subjects or some branch of them, as in the address "On the Meteorites, the Meteors, and the Shooting Stars," which he delivered in 1886 as retiring president of the American Association for the Advancement of Science, or in certain lectures in the public courses of the Sheffield Scientific School of Yale University, entitled "The story of Biela's Comet" (1874), "The relation of Meteorites to Comets" (1876), "The Worship of Meteorites" (1889), or in the articles on Meteors in the *Encyclopædia Britannica* and *Johnson's Cyclopædia*.

If we ask what traits of mind and character are indicated by these papers, the answer is not difficult. Professor Klein has divided mathematical minds into three leading classes: the logicians, whose pleasure and power lies in subtility of definition and dialectic skill; the geometers, whose power lies in the use of the space-intuitions; and the formalists, who seek to find an algorithm for every operation.† Professor Newton evidently belonged to the second of these classes, and his natural tastes seem to have found an equal gratification in the development of a system of abstract geometric truths, or

* These were detailed in a bibliography annexed to this paper in *Amer. Jour. Sci.*, ser. 4, vol. iii.

† *Lectures on Mathematics* (Evanston), p. 2.

in the investigation of the concrete phenomena of nature as they exist in space and time.

But these papers show more than the type of mind of the author; they give no uncertain testimony concerning the character of the man. In all these papers we see a love of honest work, an aversion to shams, a distrust of rash generalizations and speculations based on uncertain premises. He was never anxious to add one more guess on doubtful matters in the hope of hitting the truth, or what might pass as such for a time, but was always willing to take infinite pains in the most careful test of every theory. To these qualities was joined a modesty which forbade the pushing of his own claims, and desired no reputation except the unsought tribute of competent judges. At the close of his article on meteors in the *Encyclopædia Britannica*, which has not the least reference to himself as a contributor to the science, he remarks that "meteoric science is a structure built stone by stone by many builders." We may add that no one has done more than himself to establish the foundations of the science, and that the stones which he has laid are not likely to need relaying.

The value of Professor Newton's work has been recognized by learned societies and institutions both at home and abroad. He received the honorary degree of Doctor of Laws from the University of Michigan in 1868. He was president of the section of Mathematics and Astronomy in the American Association for the Advancement in Science in 1875, and president of the Association in 1885. On the first occasion he delivered an address entitled "A plea for the study of pure Mathematics"; on the second the address on Meteorites, etc., which we have already mentioned. Of the American Mathematical Society he was vice-president at the time of his death. In 1888 the J. Lawrence Smith gold medal was awarded to him by the National Academy for his investigations on the orbits of meteoroids. We may quote a sentence or two from his reply to the address of presentation, so characteristic are they of the man that uttered them: "To discover some new truth in nature," he said, "even though it concerns the small things in the world, gives one of the purest pleasures in human experience. It gives joy to tell others of the treasure found."

Besides the various learned societies in our own country of which he was a member, including the American Academy of Arts and Sciences from 1862, the National Academy of Sciences from its foundation in 1863, the American Philosophical Society from 1867, he was elected in 1872 Associate of the Royal Astronomical Society of London, in 1886 Foreign Fellow of the Royal Society of Edinburgh, and in 1892 Foreign Member of the Royal Society of London.

But the studies which have won for their author an honorable reputation among men of science of all countries, form only one side of the life of the man whom we are considering. Another side, probably the most important, is that in which he was identified with the organic life of the College and University with which he had been connected from a very early age. In fact, we might almost call the studies which we have been considering, the recreations of a busy life of one whose serious occupation has been that of an instructor. If from all those who have come under his instruction we should seek to learn their personal recollections of Professor Newton, we should probably find that the most universal impression made on his students was his enthusiastic love of the subject which he was teaching.

A department of the University in which he took an especial interest was the Observatory. This was placed under his direction at its organization, and although he subsequently resigned the nominal directorship, the institution remained virtually under his charge, and may be said to owe its existence in large measure to his untiring efforts and personal sacrifice in its behalf.

One sphere of activity in the Observatory was suggested by a happy accident which Professor Newton has described in the *American Journal of Science*, September, 1893. An amateur astronomer in a neighboring town, Mr. John Lewis, accidentally obtained on a stellar photograph the track of a large meteor. He announced in the newspapers that he had secured such a photograph, and requested observations from those who had seen its flight. The photographic plate and the letters received from various observers were placed in Professor Newton's hands, and were discussed in the paper mentioned. The advantages of photographic observations were so conspicuous that Professor Newton was anxious that the Observatory should employ this method of securing the tracks of meteors. With the aid of an appropriation granted by the National Academy from the income of the J. Lawrence Smith fund, a battery of cameras was mounted on an equatorial axis. By this means, a number of meteor-tracks have been obtained of the August meteors, and in one case, through a simultaneous observation by Mr. Lewis in Ansonia, Professor Newton was able to calculate the course of the meteor in the atmosphere with a probable error which he estimated at less than a mile. The results which may be expected at the now near return of the Leonids, will be of especial interest, but it will be for others to utilize them.

Professor Newton was much interested in the collection of meteorites, and the fine collection of stones and irons in the Peabody Museum of Yale University owes much to his efforts in this direction.

Professor Newton was a member of the American Metrological Society from the first, and was conspicuously active in the agitation which resulted in the enactment of the law of 1866, legalizing the use of the metric system. He prepared the table of the metric equivalents of the customary units of weights and measures which was incorporated in the act, and by which the relations of the fundamental units were defined. But he did not stop here. Appreciating the weakness of legislative enactment compared with popular sentiment, and feeling that the real battle was to be won in familiarizing the people with the metric system, he took pains to interest the makers of scales and rulers and other devices for measurement in adopting the units and graduations of the metric system, and to have the proper tables introduced into school arithmetics.

He was also an active member of the Connecticut Academy of Arts and Sciences, serving several years both as secretary and president,—also as member of the council. He was associate editor of the *American Journal of Science* from 1864, having especial charge of the department of astronomy. His notes on observations of meteors and on the progress of meteoric science, often very brief, sometimes more extended, but always well considered, were especially valuable.

In spite of his studious tastes and love of a quiet life, he did not shirk the duties of citizenship, serving a term as alderman in the city council, being elected, we may observe, in a ward of politics strongly opposed to his own.

Professor Newton married, April 14th, 1859, Anna C., daughter of the Rev. Joseph C. Stiles, D.D., of Georgia, at one time pastor of the Mercer Street Presbyterian Church in New York City, and subsequently of the South Church in New Haven. She survived her husband but three months, leaving two daughters.

In all these relations of life, the subject of this sketch exhibited the same traits of character which are seen in his published papers, the same modesty, the same conscientiousness, the same devotion to high ideals. His life was the quiet life of the scholar, ennobled by the unselfish aims of the Christian gentleman; his memory will be cherished by many friends; and so long as astronomers, while they watch the return of the Leonids marking off the passage of the centuries, shall care to turn the earlier pages of this branch of astronomy, his name will have an honorable place in the history of the science.

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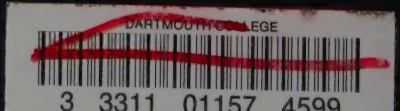
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